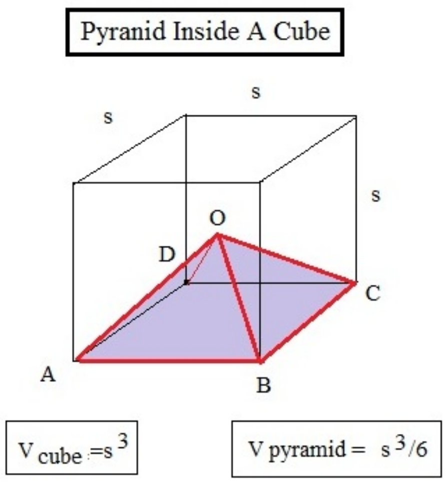
VOLUME OF AN IRREGULAR CONVEX HEXAHEDRON

The other day while having breakfast I noticed that the cantaloupe pieces I was eating had the shape of six-sided convex polyhedra with none of their sides parallel to each other. I asked myself, is it possible to calculate the volume of such pieces given the coordinates of the eight vertexes? The answer is in the affirmative as we will now show using some vector operations.

We start with a regular six-sided cube of side length s. Next we draw a square base pyramid using one of the cube sides A-B-C-D as the base. The pyramid vertex is placed at the cube center O which is located at [0,0,0] as shown-



The pyramid has volume V=(1/3)(base x height)=s^3/6 so that by symmetry there must be a total of six of these identical pyramids which make up the volume s^3 of the cube. If we draw a line AC connecting opposite corners of the base, we see that this will break the one pyramid shown into two equal volume sub-pyramids of volume s^3/12 each. We know from vector calculus that the slanted edges OB plus the base lengths AB and CB of the sub-pyramid ABCO can be expressed in vector form as

$$V_{OB} = (0-\frac{s}{2})i + (0-\frac{s}{2})j + (0+\frac{s}{2})k$$

$$V_{AB} = (-\frac{s}{2}-\frac{s}{2})j$$

$$V_{CB} = (-\frac{s}{2}-\frac{s}{2})i$$

where i, j, and k are the unit base vectors in the x, y, and z directions, respectively. Now the area of the sub-pyramid triangular base just equals s^2/2 which can also be written as the magnitude of half of the vector product between V_{AB} and $V_{CB.}$ That is-

$$\text{area} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 0 & -s & 0 \\ -s & 0 & 0 \end{vmatrix} = s^2/2$$

The height of the sub-pyramid is just h=|V_{OB}|cos(θ)=s/2. Since the volume of a pyramid is always equal to one third times its base multiplied by its height, we get that the sub-pyramid volume expressed in vector form is-

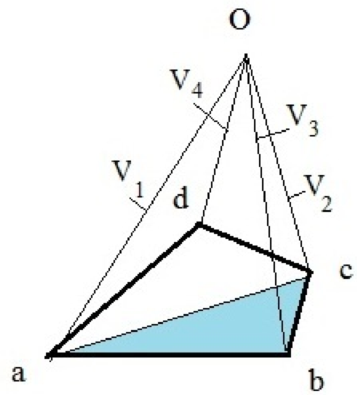
$$V_{subpyramid} = \frac{1}{6}[V_{OB} \cdot (V_{AB} x V_{CB})] = \frac{s^3}{12}$$

Here the dot refers to the dot product and the x to the cross product of the vectors shown. One can express this scalar vector product in the square bracket as the 3 by 3 determinant-

$$\begin{vmatrix} -s/2 & -s/2 & s/2 \\ 0 & -s & 0 \\ -s & 0 & 0 \end{vmatrix}$$

We are now in a position to calculate the volume of our cantaloupe pieces once the coordinates of the eight vertexes have been specified. Knowing the vertex locations we can draw eight radial lines going from them to the center at [0,0,0]. In turn we can express the slant edges of the resultant six pyramids in terms of vectors just as we did with the earlier regular cube. Let these eight vectors be V_1,V_2,V_3,V_4,V_5,V_6, V_7, and V_8. Let us use three of these vectors to define the sides of a triangular base sub-pyramid based on a pyramid whose base represents the irregular four sided area a-b-c-d as shown in the following figure-

Sub-Pyramid abcO above the Quadrangle abcd



$$\text{Vol}_{\text{sub-pyramid}} = (1/6)\{V_3 \cdot (V_1 \times V_2)\}$$

The volume of the sub-pyramid with light blue base is, in view of the earlier discussion , equal to-

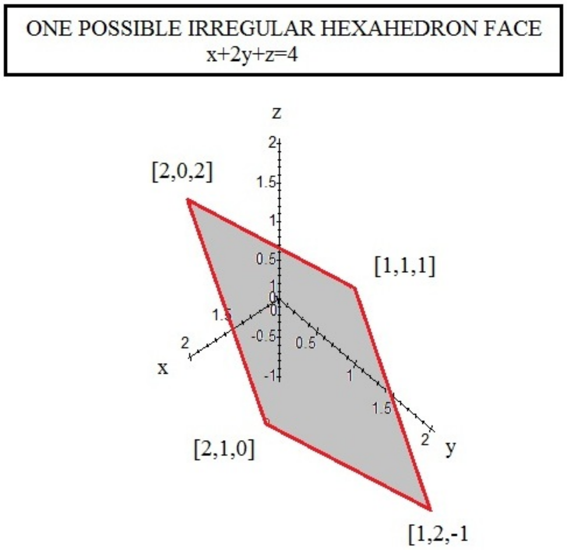
$$V_{sub-pyramid} = \frac{1}{6}|V_2 \cdot (V_1 x V_3)|$$

Thus the entire pyramid volume having abcd as its irregular four sided base is

$$V_{pyramid} = \frac{1}{6}\{|V_2 \cdot (V_1 x V_3)| + |V_4 \cdot (V_1 x V_3|\}$$

The total volume of the melon piece is then the sum of the six pyramid volumes forming this convex irregular hexahedron.

To demonstrate things for a real problem one needs to select six four sided faces which form the surface of the irregular hexahedron. As one knows a plane is defined by three points so that if a 4^{th} point is to be part of the plane then its values are restricted. If, for example , we consider the three points [1,1,1],[2,1,0],and [2,0,2] as the location of three of the corners on a given face then they will define a plane as x+2y+z=4. The 4^{th} point with locus [a,b,c] must be made to fit this equation. One possible solution of many is [a,b,c]=[1,2,-1]. The resultant 3d surface looks as follows-



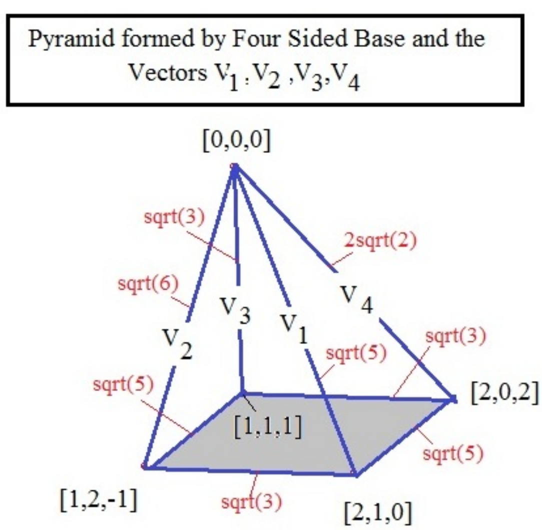
In this case the side vectors for the pyramid extending from this quadrangle face to the pyramid vertex and hexahedron center at [0,0,0] are-

$$V_1 = -2i - j, V_2 = -i - 2j + k, V_3 = -i - j - k, \text{ and } V_4 = -2i - 2k$$

On substituting into the volume equation we find-

$$V_{pyramid} = \frac{1}{6}\{|-4| + |4|\} = \frac{4}{3}$$

A graph of the pyramid follows-



Note that this pyramid is rather strange in that it has no obvious symmetries however the lengths of its edges can all be described in terms of square roots of 2, 3, 5, and 6. If we were to repeat this procedure for five more faces, contiguous to the grey face shown in the last figure, one would obtain the volume of the complete convex irregular hexahedron. The problem is thus solved and has shown that an even simple geometrical problem can have a rather complex solution.

I leave you with one last interesting and probably apocryphal story about Thomas Edison asking one of his classically trained European engineers to determine the volume of a light bulb. The assistant spent several days on the problem modeling the bulb as a combination cylinder and hemisphere and showed his approximate result to Edison rather proudly. Edison looked at the results and said I have a better way and proceeded to fill the bulb with water and then measured the volume by pouring it into a graduated beaker. The moral of the story is next time you are asked to determine the volume of a complex shaped object such as the melon piece under consideration or even more asymmetric bodies don't just blindly proceed with complicated calculations when simpler alternate solution routes such as a fluid displacement test are possible.