ITERATION PROCEDURES TO FIND VALUES OF CERTAIN CONSTANTS

The ancient Babylonians(1500 BC) found a way to calculate roots of numbers using the iteration procedure-

$$f[n+1]=\{f[n]+(N/f[n])\}/2$$

, where N is the number whose root is to be found and f[0] has a value near sqrt(N). This iteration was re-discovered by Newton-Raphson some three thousand years later using calculus. Here is the procedure. Starting with the first two terms of a Taylor expansion-

$$f[n+1]\sim f[n]+(f[n+1]^2-f[n]^2)/(2f[n])$$

, we get on replacing f[n+1]^2 by N, the identical Babylonian result. Evaluating these iterates for N=2 and f[0]=1 produces the six row table-

n	f[n]
0	1=1
1	3/2=1.5
2	17/12=1.416
3	577/408=1.414215
4	665857/470832=1.4142135623
infinity	sqrt(2)=1.41421356237309504

Notice how the iterates approach sqrt(2) at an accelerated rate as n gets large. It is truly amazing that the ancient Babylonians where able to give an accuracy of root two to eight digital places as indicated on one of their cuneiform tablets.

Besides finding the roots of integers N, iterations allow one to quickly approximate the values of various other constants. We will look at some of these below. Let us begin with the constant

$$\exp(1)=2.718281828459045235360287471352 66\ldots$$

I memorize the first 33 digits of this irrational number with a mnemonic I constructed several years ago. It reads-

2.7- followed by Andrew Jackson inauguration twice (1828- 1828)-right triangle(45-90-45)-Fibonacci three(235)-full circle(360)-year before the crash(28)-Boing jet(747)-end of black death in Europe(1352)-route west(66) .

To derive an iterative formula for this important constant, we start with the finite sum difference-

f[n+1]=f[n]+1/n!   subject to   f[0]=1

This yields f[1]=2, f[2]= 3, f[3]=8/3, f[4]= 65/24, and f[5]=163/60. As n goes to infinity the value of f[n] approaches exp(1). One can speed up the calculation for values of f[n] by using the one line computer program-

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for n from 0 to 30 do f[n+1]:=evalf(f[n]+1/(n!),30)od;
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after setting f[0]:=0. The program yields an approximation for exp(1) of-

f[27] = 2.71828182845904523536028747

Next let us look at the constant

π = 3.14159265358979323846264338328…

which represents the ratio of the circumference of a circle to its diameter. Here we have the identity-

$$\pi/4=\int_{x=0}^{1}\frac{dx}{(1+x^2)}=\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}=1-1/3+1/5-1/7+1/9$$

We can write-

f[0]=1, f[1]=2/3, f[2]=13/15, f[3]=76/105, f[4]=789/945. From it we find-

f[1]-f[0]=-1/3, f[2]-f[1]=1/5 , f[3]=f[2]=-1/7, and   f[4]-f[3]=1/9. So we have the slowly convergent iteration formula-

f[n+1)=f[n]+(-1)^(n+1)/(2n+3)   subject to   f[0]=1

which at f[100] gives the value 0.78787335 good to only two decimal places when compared with π/4=0.785398… To improve the convergence one needs to use some modified forms of arctan(x) with x<<1. One such formula is our own which reads-

π/4=12arctan(1/38)+20arctan(1/57)+7arctan(1/239)+24arctan(1/268)

This can be rewritten as-

$$\pi/4=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\left\{\frac{12}{38^{(2*k+1)}}+\frac{20}{57^{(2*k+1)}}+\frac{7}{239^{(2*k+1)}}+\frac{24}{268^{(2*k+1)}}\right\}$$

Here we don't have to go through an iteration procedure as π/4 is given directly by the indicated infinite sum. Summing up to just k=20 we get the 68 place accurate approximation-

π/4~0.78539816339744830961566084581987572104929234984377645 5695

An interesting mnemonic for π is the following-

MNEMONIC FOR THE FIRST 15 DIGITS OF PI

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9
How	I	want	a	drink,	alcoholic	of	course,	after	the	heavy	lectures	involving	quantum	mechanics

Pi= 3.14159265358979

Consider next the value of the Golden Ratio ϕ=(1+sqrt(5))/2. Here we start with the identity (sqrt(5)-2)(sqrt(5)+2)=1 to get the iteration formula-

f[n+1]=3/2+1/(4f[n])   subject to f[0]=1

This produces f[1]=7/4, f[2]=23/14, f[3]=38/23, f[4]=251/156 with-

ϕ=f[∞]=1.61803398874989484820458683436563811772…

Next consider the Euler Constant defined as-

$\gamma = \frac{lim}{n\to\infty}$ {1+1/2+1/3+….1/n -ln(n)= 0.5772156649015328606065120900824… The appropriate corresponding iteration formula is-

f[n+1]=f[n]+1/(n+1)+ln(n/(n+1)) subject to f[1]=1

It yields f[2]=3/2-ln(2), f[3]=11/6-ln(3), and f[4]=25/12-ln(4). The convergence rate is here seen to be very slow. At f[1000] one finds 0.577715582 so only three places of accuracy. The iterative term we are using is here -

f[n] := Psi(n + 1) + gamma - ln(n)

As a final constant consider ln(2)=0.69314718...Carrying out a two term series expansion we find-

ln(1+x)≈ln(1+a)+(x-a)/(1+a)

On setting x=1 with f[n+1]=ln(1+x) and f[n]=(1+a) , we get-f[n+1]=f[n]-1+exp(f[n+1])/exp(f[n])

Next letting f[1]=1 and replacing exp(f[n+1])by 2, we get the iterative formula-

f[n+1]=f[n]-1+2/exp(f[n])   with the initial condition f[1]=1.

We find f[2]=1-1+2/exp(1)=2/exp(1)=0.73675 and f[3]=2/exp(1)-1+2/exp(2/e)=0.69404.This second iteration f[3] is already accurate to two places. A seventh iteration f[7] produces the ninety digit accurate result-

ln(2)=

0.69314718055994530941723212145817656807550013436025525412068000949339362196969471560586332 7

There are numerous other iteration formulas capable of generating constants to any desired order of accuracy. Among these are the trigonometric functions which posses rapidly convergent formulas. Thus, for example, cos(θ) at one radian can be quickly generated by –

f[n+1]=f[n]+(-1)^n/(2n)! subject to f[1]=1.

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