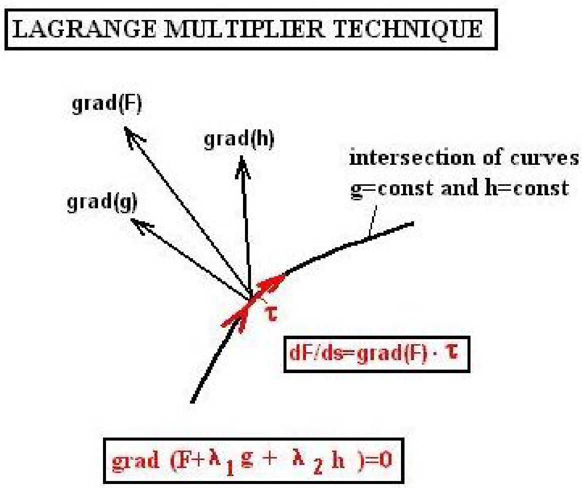
LAGRANGE MULTIPLIERS

In our above variational methods course we briefly discussed Lagrange Multipliers and showed how these may be used to find the extremum of a function F subject to a set of constraints. We want to here discuss this procedure in more detail and work out several more specific examples of possible interest to the readers. Consider a function of n variables given as-

$F(x_1,x_2,x_3,\ldots x_n)$ plus constraints $g_1(x_1,x_2,x_3,\ldots x_n)=c_1$, $g_2((x_1,x_2,x_3,\ldots x_n)=c_2$,..etc

where the c_ns are constants. Geometrically one can think of $g_n=c_n$ as hypersurfaces which intersect in a common curve C . The gradients of the various g surfaces will be at right angles to the intersection curve as shown-



Here g_1=g and g_2=h in order to simplify the discussion. Note next that the directional derivative of the function F to be extremized (and hence have dF/ds=0) is-

$$\frac{dF}{ds} = \nabla F \cdot \tau \ \textit{where } \tau \textit{ is the unit length} \tan \textit{gent vector along curve } C$$

Thus we have that the gradient of F is also perpendicular to curve C and hence that grad(g), grad(h) and grad(F) are coplanar. This means in general that-

$$\nabla\left[F + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \ldots.\right] = 0$$

where the λs are the Lagrange Muliplers. These n equations plus the equations for the constraints constitute sufficient information to find all λs plus determine the extremum value for F.

Lets consider a few examples starting with the simple problem of determining the maximum volume V contained in a cylinder of radius R and height H for fixed surface area S . Here one has-

$$\nabla[\pi R^2 H + \lambda_1(2\pi R^2 + 2\pi RH)] = 0$$

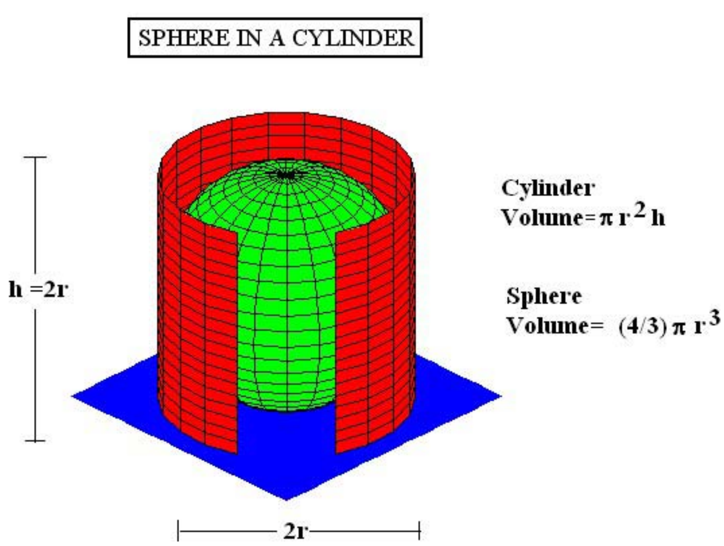
Thus one finds the following three equations-

$$RH + \lambda_1(2R + H) = 0 \,,\, R + 2\lambda_1 = 0 \,,\, S = 2\pi R(R + H)$$

with solutions -

$$\lambda_1 = -R/2 \,,\, H = 2R \,,\, S = 6\pi R^2$$

The result states that for maximum possible storage a can should have its diameter just equal to its height. There is an interesting experiment carried our by child psychologists in which they fill both a tall and a wide drinking glass full of the same volume of fruit juice and then ask a child which glass contains the larger amount of fluid. Invariably the child will choose the taller glass. Directly related to this maximum volume problem of a cylinder is the Archimedes observation that the maximum volume sphere which can be put in a cylinder requires that the cylinder height just equals its diameter. Under those conditions one has that the sphere to cylinder volume is exactly 2/3. A graph of this geometry(and one related to that supposedly engraved on Archimedes's tombstone) is-



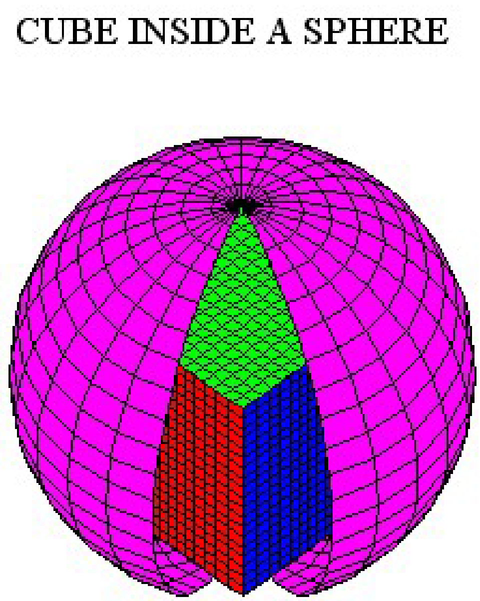
Next let us ask what is the radius of the largest volume rectangular solid which can fit into a unit radius(R=1) sphere. Here the Lagrange Multiplier method produces-

$$\nabla [xyz + \lambda_1 (x^2 + y^2 + z^2) = 0$$

or the equivalent-

$$yz + 2\lambda_1 (x) = 0 \, , \, xz + 2\lambda_1 (y) = 0 \, , \, and \;\, xy + 2\lambda_1 (z) = 0$$

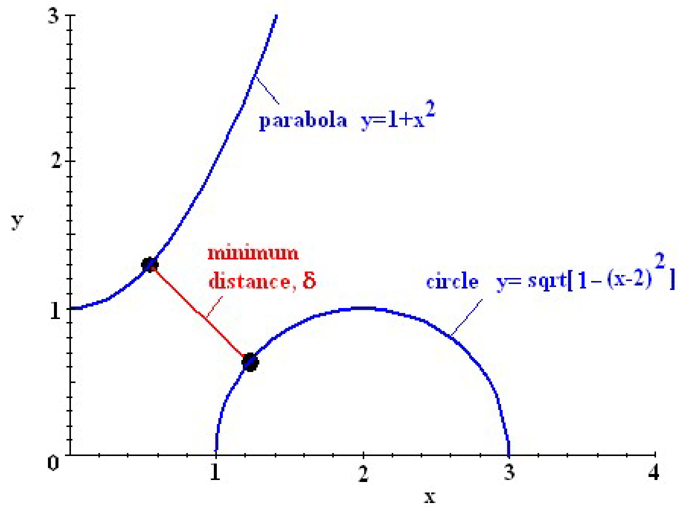
These yield the solutions x=y=z=1/sqrt(3). That is, the largest volume rectangular solid capable of fitting into a unit radius sphere is a cube with sides of length 2/sqrt(3). We show you here a 3D picture of this result constructed via MAPLE-



Note that the ratio of cube volume V_c=8x^3 to that of the sphere volume Vs=4π/3 is 2/[πsqrt(3)]=0.3675.. .

As a third, more difficult, example, consider the shortest distance from the parabola y=1+$x^2$and the circle $(x-2)^2$+y^2=1=0. This time one wants to extremize the distance squared between the two curves subjected to the curve constraints. One has the following picture-

MINIMUM DISTANCE BETWEEN TWO CURVES



We find-

$$\nabla [\,(x_1 - x_2)^2 + (y_1 - y_2)^2 + \lambda_1(1 + x_1^{\,2} - y_1) + \lambda_2((x_2 - 2)^2 + y_2^{\,2} - 1))] = 0$$

which produces the four equations-

$$2(x_1 - x_2) + 2x_1\lambda_1 = 0\,,\ -2(x_1 - x_2) + 2\lambda_2(x_2 - 2) = 0\,,\ 2(y_1 - y_2) - \lambda_1 = 0\,,$$
$$and\ -2(y_1 - y_2) + 2\lambda_2 y_2 = 0$$

These, when used in conjunction with the constraints $y_1=1+x_1^2$ and $(x_2-2)^2+y_2^2=1$, yields, after elimination of the λs and y_1 and y_2 , the rather nasty set of two highly non-linear algebraic equations-

$$2x_1[(1 + x_1^{\,2}) - \sqrt{1 - (x_2 - 2)^2} + (x_1 - x_2) = 0 \quad and$$

$$(x_1 - x_2)\sqrt{1 - (x_2 - 2)^2} - (x_2 - x_1)[(1 + x_1^{\,2}) - \sqrt{1 - (x_2 - 2)^2} = 0$$

They can be solved graphically and one finds –

$$x_1 = 0.5536..,\quad x_2 = 1.2579..,\quad y_1 = 1.30640..,\ and\ y_2 = 0.67029..$$

The results also show that the minimum distance between the curves will be-

$$\delta = 0.94905..$$

Note from the above figure that this shortest route between the constraint curves occurs where their slopes are equal , a fact making possible a rapid evaluation of the above equations since the figure allows one to make good initial estimates for the expected values.

As a final example of a Lagrange Multiplier application consider the problem of finding the particular triangle of sides a, b, and c whose area is maximum when its perimeter L=a+b+c is fixed. Our starting point here is Heron's famous formula for the area of a triangle-

$$A=\sqrt{s(s-a)(s-b)(s-c)}\ \ \textit{where}\ s=(a+b+c)/2=L/2\ \textit{as the half perimeter}$$

A little manipulation allows one to recast this result in the form-

$$A^2=\frac{[(a+b)^2-c^2][c^2-(a-b)^2]}{16}$$

Next applying the Lagrange Multiplier method we have-

$$\nabla\left[[(x+y)^2-z^2][z^2-(x-y)^2]+\lambda[x+y+z]\right]=0$$

or the equivalent three algebraic expressions-

$$2(x+y)[z^2-(x-y)^2]-[(x+y)^2-z^2]2(x-y)+\lambda=0,$$
$$2(x+y)[z^2-(x-y)^2]+[(x+y)^2-z^2]2(x-y)+\lambda=0,$$
$$-2z[z^2-(x-y)^2]+[(x+y)^2-z^2]2z+\lambda=0$$

Eliminating λ one has (x+y)U-V(x-y)=(x+y)U+V(x-y)=-zU+Vz with $U=z^2-(x-y)^2$ and $V=(x+y)^2-z^2$. Combining the first two we find 2(x-y)V=0 so that x=y. Combining the second and the last one has z(V-U)=2xU or(z+2x)(z-x)=0 implying that z=x=y. Thus one may conclude that the triangle of largest area subjected to the constraint of a fixed perimeter is an equilateral triangle. Using the Heron area formula, one has-

$$A=\frac{1}{4}\sqrt{[3a^2][a^2]}=\frac{a^2\sqrt{3}}{4}$$

for the largest possible area triangle for fixed perimeter L=3a.

June 2009