LATEST ON THE NUMBER FRACTION

About a decade ago, while studying number theory, we came up with a new quotient defined as-

f(N)=[sum of all factors of N not including the end values of N and 1]/N

In terms of the sigma function in Number Theory for N, this definition reads-

$$f(N)=[\sigma(N)-N-1]/N$$

We have termed this quotient the **Number Fraction** . It has numerous properties some of which will be discussed below.

We begin by noting that if N equals a prime number p, its number fraction f(p) will always be zero. For composite Ns, this function will have a unique fractional value greater than zero. Thus f(6)=2+3)/6=5/6 and f(2431)=592/2431. When f(N) gets much above one it has an unusually large number of devisors and is designated by us as a **super-composite**. An example of such a super-composite is-

$$N:=174636000 \text{ where } f(N)=638768591/174636000=3.65771428\ldots$$

The simplest evaluations of f(N) occurs for N=p^n, where p is any prime number. We have-

$$f(p)=0$$

$$f(p^2)=1/p$$

$$f(p^3)=(1+p)/p^2$$

Generalizing, we get-

$$f(p^n)=[1+p+p^2+\ldots+p^{(n-2)}]/[p^{(n-1)}]$$

which, after some further manipulations, produces the general formula-

$$f(p^n)=[1-p^{(1-n)}]/[p-1]$$

for any prime p and integer power $n \geq 1$. So we have, for p=71 and n=3 the result-

$$f(71^3)=f(357911)=[1-71^{(-2)}]/[70]=72/5041$$

The four divisors of N=357911 are {1, 71, 5041, 357911}, so that σ(N)=363024 and f(N)=(71+5041)/N

Here, for later reference, N can also be considered as a semi-prime with prime components p=71 and q=5041.

Note from the $f(p^n)$ formula, we can recover the very simple results that $f(p^2)=1/p$ and

$f(2^n)=1-[1/2^{(n-1)}]$. Also one has $f(p^n)=1/(p-1)$ in the limit of n becoming infinite.

Note that if N is a product of several primes taken to specified powers, the evaluation of f(N) becomes more complicated. Here the starting point.

In any evaluation of f(N) not covered explicitly by the above formulas, one starts with the identity-

$$N=(p_1^a)(p_2^b)(p_3^c)\ldots \quad \text{with p primes and a, b, c,.. integer exponents}$$

Working out two cases where the exponents are one, we have-

$$f(pq)=[p+q]/pq \qquad \text{and} \qquad f(pqr)=[p+q+r+(pr+pq+rq)]/pqr$$

The more primes present in N the more difficult the general form for f(N) becomes. For f(3*5*7)=f(105) we get –

$$f(105)=[3+5+7+(15+21+35)]/105=86/105$$

When N=pq we are dealing with a standard semi-prime. An example is f(77)=f(7*11)=(7+11)/77 =18/77 . Such semi-primes, when p and q get very large, play an important role involving public keys in cryptography. For such semi-primes we have-

$$Nf(N)=p+q \quad \text{and} \quad N=pq$$

Eliminating either p or q we find that the prime factors of N are-

$$[p,,q]=H\mp \mathrm{sqrt}(H^2-N) \quad \text{with } H=Nf(N)/2$$

Since most advanced mathematics computer programs, such as Maple or Mathematica, give σ(N) to at least 40 places, the value of f(N) is readily established using the identity-

$$Nf(N)=\sigma(N)-N-1$$

Consider factoring the semi-prime-

$$N=290212357367 \quad \text{for which sigma}(N)=290213609496 \quad \text{and } H=Nf(N)/2=626064$$

For this, the [p,q] formula above yields-

$$[p,q]=[307091, 945037]$$

Multiplying p by q produces N. This confirms our factors.

Finally we want to look at super-composites. We find on plotting f(N) for any integer versus N that there are certain Ns for which f(N) becomes considerably larger than its immediate neighbors. These are what we refer to as **super-composites**. Typical super-composites have an ifactor of the form-

$$N=2^a*3^b*5^c*\ldots \quad \text{with } a>b>c$$

That is, they contain the highest powers for the lowest primes. An example is-

N=1244160=2^10*3^5*5

Applying the Maple computer program-

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listplot([seq([t,(sigma(t)-t-1)/t],t=1244155..1244165)],thickness=2,color=red);
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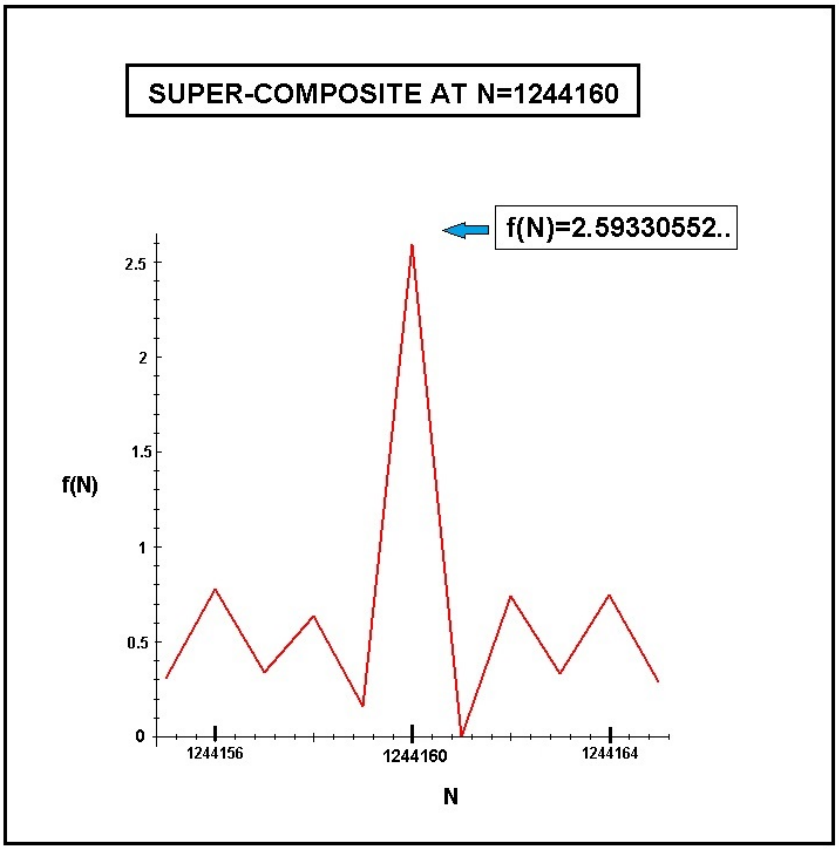
, we get the following graph-



The graph clearly shows how a super-composite towers above its immediate neighbors. Often moving just one unit away from a super-composite will produce a prime number. In the present case 1244161 factors into 271 x 4591. This means its a semi-prime. Here f(1244161)=0.00390016. Such a small value for f when N is a semi-prime is expected.

Finally we look at f(n!) and ifactor(n!) values for the factorial n!. Here we get the table-

n	n!	f (n!)	ifactor(n!)
2	2	0	2
3	6	0.833333333	2·3

4	24	1.458333333	2^3·3
5	120	1.991666667	2^3·3·5
6	720	2.356944444	2^4·3^2·5·7
7	5040	2.837896825	2^4·3^2·5·7
8	40320	2.946403770	2^7·3^2·5·7
9	162880	3.081346451	22
10	1628800	3.225663305	2^7·5^2·509

It is seen that these factorials have f(n!) values above one for n=4 and above. The measure of super-compositeness increases with increasing n. For n=100 we get-

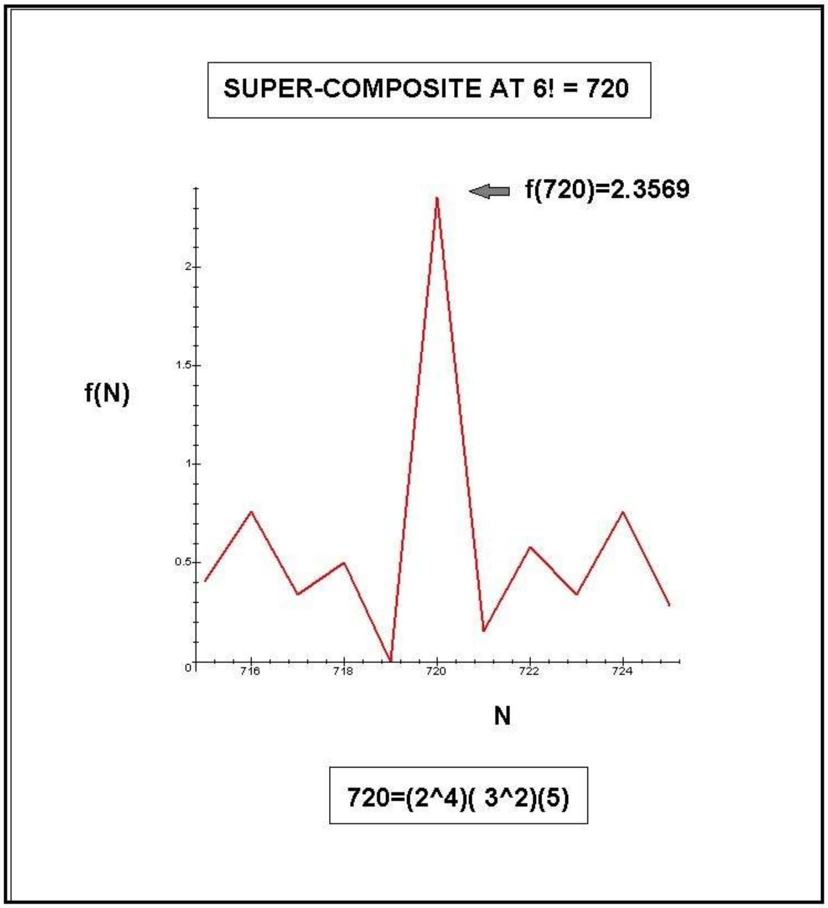
f(100!)= 7.293771354 with ifactor(100!)= 2^97·3^48·5^24 · {plus another 22 primes with decreasing powers powers).

Note that the powers of the lowest three primes 97-48-24 go approximately as 4-2-1. This type of ratio continues to hold for even lager values of n!. Thus ifactor(1000!) starts as (2^994)(3^498)(5^249) and its f(1000!) value goes as 11.3491644.. .

These results allow us to introduce a new unique number –

N=2^4a)(3^2a)(5^a) for a=1,2,3,4,

For a=1, one has N=720=6!=(2^4)·(3^2)·5. A plot of f(N) in its neighborhood of 6! follows -



We find that often the number N!±1 will be a prime number . You notice this at (6!-1)=719. Other primes are found at 3!-1, 4!-1, 6!-1, 12!-1, 14!-1, 27!+1, 30!-1, 32!-1, 33!-1, 37!+1, 38!-1, 41!+1, 73!+1. Similar to the Mersenne Primes we have here a new set of primes given by-

$$P=N!\pm1$$

It is assumed that there are an infinite number of these although the spacing between neighboring Ps can become large. From earlier notes we know that all primes above five have (P!) mod(6)=1 or 5. This means that not just the P rimes but any other primes are also diviseble by either 6n+1 or 6n-1. It will also always be the case that any larger prime must have f(prime)=0

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