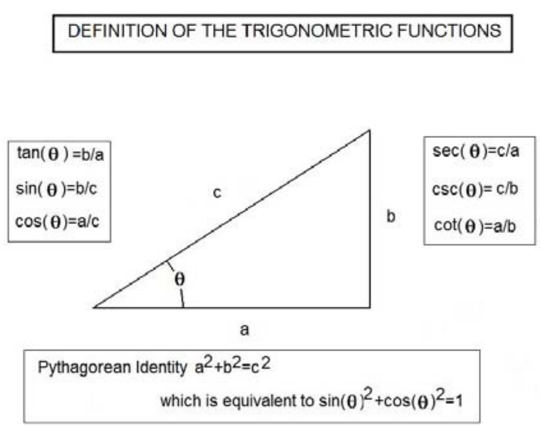
GENERATING HIGHLY ACCURATE VALUES FOR ALL TRIGONOMETRIC FUNCTIONS

INTRODUCTION:

Although hand calculators and electronic computers have pretty much eliminated the need for trigonometric tables starting about sixty years ago, this was not the case back during WWII when highly accurate trigonometric tables were vital for the calculation of projectile trajectories. At that time a good deal of the war effort in mathematics was devoted to building ever more accurate trig tables some ranging up to twenty digit numerical accuracy. Today such efforts can be considered to have been a waste of time, although they were not so at the time.
It is our purpose in this note to demonstrate a new approach for quickly obtaining values for all trigonometric functions exceeding anything your hand calculator can achieve.

BASIC TRIGNOMETRIC FUNCTIONS:

We begin by defining all six basic trigonometric functions by looking at the following right triangle-



The sides a,b, and c of this triangle satisfy the Pythagorean Theorem-

$$a^2+b^2=c^2$$

we define the six trig functions in terms of this triangle as-

$\sin(\theta)=b/c$ $\cos(\theta)=a/c$ $\tan(\theta)=b/a$

$\csc(\theta)=c/b$ $\sec(\theta)=c/a)$ $\cot(\theta)=b/a$

From Pythagorous we also have at once that –

$$\sin(\theta)^2+\cos(\theta)^2=1 \qquad and \qquad [1+\tan(\theta)^2]=\frac{1}{\cos(\theta)^2} \quad .$$

It was the ancient Egyptian pyramid builders who first realized that the most important of these trigonometric functions is tan(θ). Their measure of this tangent was given in terms of sekels, where 1 sekhel=7θ. Converting each of the above trig functions into functions of tan(θ) yields-

$$\sin(\theta)=\frac{\tan(\theta)}{\sqrt{1+\tan(\theta)^2}} \quad \cos(\theta)=\frac{1}{\sqrt{1+\tan(\theta)^2}} \quad \tan(\theta)=\tan(\theta)$$

$$\csc(\theta)=\frac{\sqrt{1+\tan(\theta)^2}}{\tan(\theta)} \quad \sec(\theta)=\sqrt{1+\tan(\theta)^2} \quad \cot(\theta)=\frac{1}{\tan(\theta)}$$

In using these equivalent definitions one must be careful of signs and infinities appearing in the approximations to tan(θ).

For all purposes it will be sufficient to examine the values of the above functions only over the restricted range-

$$-\pi/4<\theta<\pi/4$$

Since the values outside the range can readily be determined from the values existing in this range by use of the following extension equations-

$$\tan(\frac{\pi}{4}+\theta)=\frac{1+\tan(\theta)}{1-\tan(\theta)} \quad and \quad \tan(2\theta)=\frac{2\tan(\theta)}{1-\tan(\theta)^2}$$

So, for example, tan(π/4)=1 says tan(π/2)=infinite, and tan(π/3)=1/tan(π/6)=sqrt(3).
Also we have tan(π/8)= sqrt(2)-1.

APPROXIMATING TAN(θ) TO HIGH ACCURACY:

Havin shown that all trigonometric functions can be expressed in terms of tan(θ), we next seek an algorithm which will produce highly accurate values for tan(θ) in $0\le\theta\le\pi/4$. There are many ways to carry out such evaluations starting with the simplest approach of using the infinite series-

$$\tan(\theta) = \sum_{n=1}^{\infty}\{\frac{(-1)^{n-1}2^{2n}(2^{2n-1}-1)B_{2n}}{(2n)!}\}\theta^{2n-1}$$

, where B_{2n} are the even Bernoulli numbers . This approach however is quite slow in convergence when θ gets larger . Instead people have used in the past polynomial approximations of tan(θ)/θ using a polynomial in even powers of θ up through the 12th power. This approach yields reasonable values but still lacks the accuracy of a much more recent method first developed by us some six years ago and based on the use of Legendre polynomials.

Here is how this procedure works. We start with the integral-

$$\int_{t=0}^{1} P(2n,t)\cos(at)dt = M(n,a)\sin(a)-N(n,a)\cos(a)$$

, where M(n,a) and N(n,a) are polynomials which increase in size as n increases. We also note that the integral on the left approaches zero rapidly as the order of even Legendre Polynomial increases. This is due to the oscillatory nature of the integrand. It means for larger n we will have the approximation-

$$\tan(a) \approx T(n,a) = \frac{N(n,a)}{M(n,a)}$$

Working out the first five quotients we find-

$$T(1,a)=\frac{3a}{3-a^2}$$

$$T(2,a)=\frac{105a-10a^3}{105-45a^2+a^4}$$

$$T(3,a)=\frac{10395a-1260a^3+21a^5}{10395-4725a^2+210a^4-a^6}$$

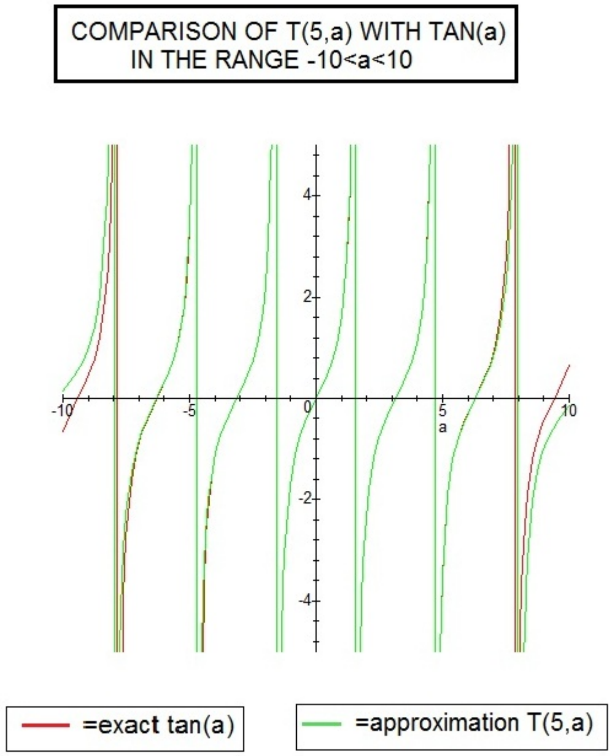
$$T(4,a)=\frac{2027025a-270270a^3+6930a^5-36a^7}{2027025-945945a^2+51975a^4-630a^6+a^8}$$

$$\mathrm{T}(5,\mathrm{a})=\frac{654729075\mathrm{a}-91891800\mathrm{a}^3+2837835a^5-25740a^7+55a^9}{654729075-310134825a^2+18918900a^4-315315a^6+1485a^8-a^{10}}$$

Each of these yields estimates for tan(a) with the accuracy increasing with increasing n and decreasing a. For a=1 rad =57.2957deg, we find T(1,1)=1.5, T(2,1)=1.557, T(3,1)=1.55740772, T(4,1)= 1.557407724 and T(5,1)= 1.557407724654902230. This should be compared to the exact value tan(1)= 1.5574077246549022305… So T(5,1) is accurate to 18 decimal places. Since we are only interested in finding tan(a) for | a|$\leq \pi/4=0.785398..$, we are guaranteed that T(5,a) will generate at least 18 digit long values for all six trigonometric functions defined above..

A GRAPH AND SOME SPECIFIC VALUES FOR T(5,a):

We begin by grsaphing T(5,a) and tan(a) over the range -10<a<10.Here are the results-



We see there is excellent agreement of tan(a) with our approximation T(5,a) as long as $|3\pi/2|>a$. The closer one gets to a=0 the more accurate the approximation becomes. Note by setting the denominator of T(5,a) to zero we find poles at ±1.57079632679490 . This is very close to the exact infinity of tan(a) at x=±1.570796326794896… We also have the approximation-

$$\pi \approx 2(1.570796326794)=3.141592653588$$

which is accurate to twelve places. A MacLaurin expansion of T(5,a) about a=0 agrees exactly with the series for tan(a) through the 19^{th} power of a.
Since tan(a) and T(5,a) are both odd functions it will be sufficient to just concern ourselves with approximations to the tangent function over the half range $0<a<\pi/4$. We have used the following program to quickly construct a table at five degree intervals between 0 and 45deg-

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a:=n*Pi/36;    for n from 0 to 9 do {(n, evalf(T(5,a),20)}od;
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Here is the table-

a in Deg.	Tangent Approx., T(5,a)	Exact Value, Tan(a)
0	0	0

5	0.087488663525924005222018	0.0874886635259240052220186
10	0.17632698070846497347109	0.176326980708464973471090
15	0.26794919243112270647255	0.267949192431122706472553
20	0.36397023426620236135104	0.363970234266202361351047
25	0.46630765815499859283000	0.466307658154998592830007
30	0.57735026918962576450914	0.577350269189625764509149
35	0.7002075382097097794585	0.700207538209709779458520
40	0.839099631177280011763	0.839099631177280011763125
45	0.99999999999999999999	1.000000000000000000000000

The T(5,a) approximation is seen to be accurate to at least twenty places over the entire range of 'a' considered. We cut-off the value of T(5,a) where a departure from tan(a) was first noted. As seen, the accuracy drops somewhat as 'a' increases.

EVAUTION OF OTHER TRIGONOMETRIC FUNCTIONS USING THE ABOVE TABLE:

Let us next determine the values of the remaining trigonometric functions by making use of the above table. We will restrict ourselves to just a=π/6 rad=30eg knowing the values for other angles can be obtained in a like manner.

We have to twenty place accuracy–

$$\sin(\pi/6) = T(5,\pi/6)/\sqrt{1+T(5,\pi/6)^2} \quad , \quad \cos(\pi/6) = 1/\sqrt{1+T(5,\pi/6)^2}$$

$$\cot(\pi/6) = 1/T(5,\pi/6) \quad , \quad \sec(\pi/6) = \sqrt{1+T(5,\pi/6)^2} \quad \textit{and}$$

$$\csc(T(5,\pi/6) = \sqrt{1+T(5,\pi/6)^2}\,/\,T(5,\pi/6)$$

Although these values are not affected by the singularities of T(5,a) , when evaluating things outside the present range on 'a' will require some consideration of these singularities . In particular the plots of sine and cosine over a wider range exceeding a=π/2 will indicate discontinuities where there are none.

In looking at the right triangle given at the beginning of this article one can set the base length to sqrt(3), the hypotenuse to 2 and the vertical side-length to 1. For this triangle the sin (θ)= ½ , cos(θ)= sqrt(3)/2 , tan(θ)=1/sqrt(3), cot(θ)=sqrt(3), sec(θ)=2/sqrt(3), and csc(θ)=2. In each of these cases the angle θ is exactly 30 deg=π/6 rad. We can use this information, among other things, to estimate the value of π as follows-

$$\pi = 6\arcsin(\frac{1}{2}) = 3 + \frac{1}{8} + \frac{9}{640} + \frac{15}{7168} + ...$$

Also we have-

$$\sqrt{3} \approx \frac{1}{T(5,\pi/6)} = 1.7320508075688772935$$

CONCLUDING REMARKS:

We have used a technique based on Legendre polynomials to evaluate the basic trigonometric functions to twenty place accuracy in the range 0<θ<π/4 radians. It is shown that all that is needed is to find accurate values of tan(θ) in this half-range is to obtain values for all other trigonometric functions at any angle using appropriate transformation and extension formulas. For most cases very accurate results (twenty decimal places or better) for tangent are obtained using an integral involving the tenth order Legendre polynomial-

$$P(10,x)=\frac{46189x^{10} - 109395x^{8} + 90090x^{6} - 30030x^{4} + 3465x^{2} - 63}{256}$$

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Gainesville, Florida