USING LEGENDRE POLYNOMIALS TO OBTAIN APPROXIMATIONS FOR CERTAIN SLOWLY VARYING FUNCTIONS OF X

About a decade ago while examining the properties of certain definite integrals containing Legendre polynomials, we noticed that slowly varying functions f(a,x) when multiplied by the even Legendre polynomials P[2n,x] yield integral values close to zero whenever n becomes large. Thus for example-

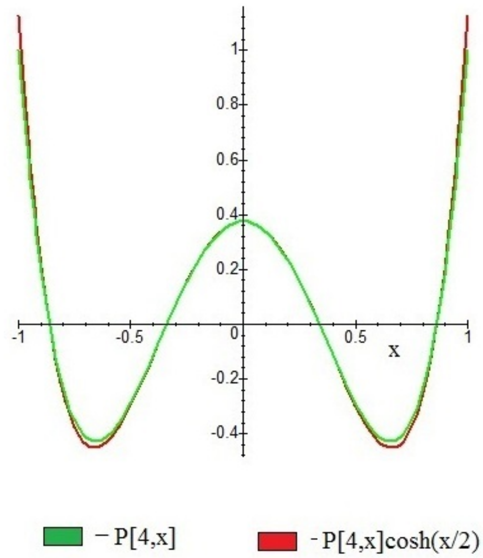
$$\int_{x=0}^{1} P[4,n]\cosh(x/2)dx = 0.00006689$$

The reason for this is the fact that Legendre polynomials are oscillatory functions with n zeros in the range -1<x<1 plus have the important property that-

$$\int_{x=0}^{1} P[2n,x]dx = 0$$

So if the function of f(x,a) multiplying P(2n,x)varies slowly with x the neighboring portions of the integral product cancel each other. This can be seen more clearly in the following plot of P[4,x] and P[4,n)cosh(x/2)-

COMPARISON OF THE INTEGRAND P[4,x] AND THE PRODUCT P[4,x]COSH(X/2) IN -1<X<1



The larger n becomes, the more accurate the comparison.

If we now take the definite integral and expand it, we have-

$$K(a,n)=\int_{x=0}^{1}P[2n,x]f(a,x)dx=h(a)[M(a,n)+N(a,n)g(a)]\approx 0$$

This produces the important new approximation-

$$g(a)\approx\frac{-M(n,a)}{N(n,a)}$$

With n being the chosen order of the approximation.

EXPONENTIAL FUNCTION EXP(a):

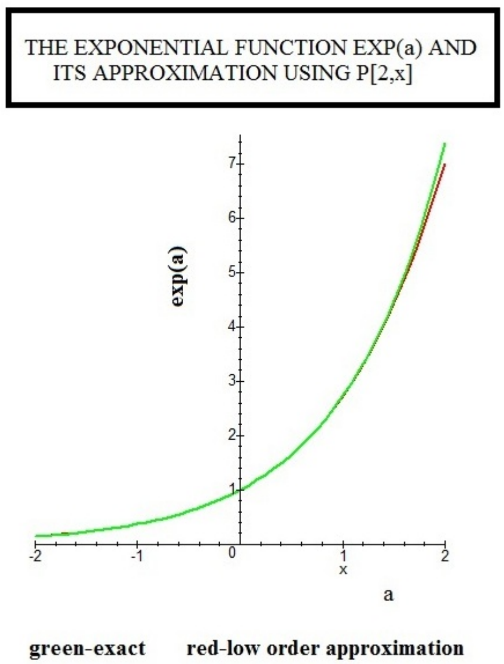
One of the simpler integrals which yields an approximation for the function exp(a) using the low order even Legendre polynomial P[2,x] is-

$$\int_{x-0}^{1} P[2,x]\cosh(ax/2)dx \approx 0 = \frac{1}{3}\left\{[a^2-6a+12]\exp(a/2)-[a^2+6a+12]\exp(-a/2)\right\}$$

It produces the n=1 approximation-

$$\exp(a) \approx \frac{[a^2+6a+12]}{[a^2-6a+12]}$$

A plot of this approximation and the exact function exp(a) follows-



This graph shows that our n=1 approximation already yields good values for exp(a) when a is kept small but does not agree to ahigh number of decimal places as ‘a’ gets large. This makes sense since the variation in cosh(ax/2) gets larger as ‘a’ increases.

Using an n=10 approximation for a=1 yields the much more accurate estimate-

$$\exp(1) \approx \frac{551157654494325100219720823521}{202759569932735203392750534601}$$

This quotient yields a value of e good to an amazing sixty places.

APPROXIMATIONS FOR ARCTAN(a) and Pi:

Another simple function for which we can generate a rapid approximation , using the present Legendre polynomial approach, is-

$$\frac{1}{a}\arctan(\frac{1}{a}) = \int_{x=0}^{1} \frac{dx}{a^2+x^2}$$

Here we have-

$$K(a,n) = \int_{x=0}^{1} \frac{P(2n,x)}{a^2+x^2}dx = -M(n,a) + N(n,a)\left[(\frac{1}{a})\arctan(\frac{1}{a}\right] \approx 0$$

For this type of polynomial quotient we can make use of the quo and rem operations in our Maple computer program to quickly evaluate M(n,a) and N(n,a).The values are-

M(n,a)=int(quo(P(2n,x),x^2+a^2,x),x=0..1)

and-

N(n,a)=rem(P(2n,x),a^2+x^2,x)

For n=4, these produce the approximation-

$$\arctan(\frac{1}{a}) \approx \left(\frac{a}{35}\right)\left\{\frac{15158+147455a^2+345345a^4+225225a^6}{35+1260a^2+6930a^4+12012a^6+6435a^8}\right\}$$

The approximation is quite accurate to multiple decimal places when ‘a’>1. However it is not very accurate at a=1 corresponding to tan(π/4). One can get around this impediment by use of certain arctan formulas involving large ‘a’. For example, if we employ our own four term arctan formula-

$$\pi = 48\arctan(\frac{1}{38}) + 80\arctan(\frac{1}{57}) + 28\arctan(\frac{1}{239}) + 96\arctan(\frac{1}{268})$$

we find , in terms of the above n=4 approximation, that-

$\pi \approx 3.14159265358979323846264338327$

This value is accurate to thirty digits. Further improvements are possible by increasing the value of n.

One notices that departures from arctan(1/a) become large for a<1. To get arctan(1/a) for these lower values of 'a', one can invoke the identity-

$$\arctan(a)+\arctan(1/a)=\pi/2$$

APPROXIMATIONS TO LN[1+(1/a)]:

Another function whose approximations are easily established by the present Legendre polynomial approach is-

$$\ln(1+a)-\ln(a)=\int_{x=0}^{1}\frac{dx}{a+x}$$

Here we have-

$$K(n,a)=\int_{x=0}^{1}\frac{P(2n,x)}{(a+x)}dx\approx 0=-M(n,a)+N(n,a)\ln(1+\frac{1}{a})$$

with the polynomials M and N given by the quo and rem operations in our computer program. For n=4 we get the approximation-

$$\ln(1+\frac{1}{a})\approx$$

$$\left(\frac{1}{280}\right)\left\{\frac{26635+1179640a^3-121272a-2762760a^5-429660a^2+1801800a^7 \backslash \\ +1231230a^4-900900a^6}{35-1260a^2+6930a^4-12012a^6+6435a^8}\right\}$$

which yields good results when a>>1. For example ln(1.01) has a=10 . Here the n=4 approximation yields –

ln(1.1)≈0.0953101798043

which is accurate to thirteen decimal places. When a<1 the present approximation breaks down .To get around this difficulty one must first use some extra properties of natural logarithms. For example we can write-

ln(2)=2ln(1.2)-ln(0.8)-ln(0.9)

so that the logarithm terms on the right are all near unity. This means we need only evaluate our n=4 approximation for a=5, -5, and -10 which all lie in the valid convergence range.

Doing so yields the eleven digit accurate result-

ln(2)=0.69314718055

We could also use the identity ln(2)=ln(1+1/3)+ln(1+1/2) where a=3 or 2. For n=4 this produces a ln(2) approximation good to six places.

APPROXIMATIONS FOR TAN(a), SIN(a), and COS(a):

The function f(x,a) multiplying P(2n,x) can also be oscillatory provided the spacing of its zeros exceeds those of the P(2n,x) being used. This means we may also consider trigonometric functions such as cos(ax). There we find-

$$K(n,a)=\int_{x=0}^{1} P(2n,x)\cos(ax)dx=[N(a,n)\sin(a)+M(n,a)\cos(a)]\approx 0$$

So we have the nth order approximation for tangent of ‘a’ given by-

$$\tan(a)\approx T(n,a)=-\frac{M(n,a)}{N(n,a)}$$

From it we also have the approximations for cosine and sine given by-

$$C(n,a)=\frac{1}{\sqrt{1+T(n,a)}}\quad and \quad S(n,a)=\frac{T(n,a)}{\sqrt{1+T(n,a)^2}}$$

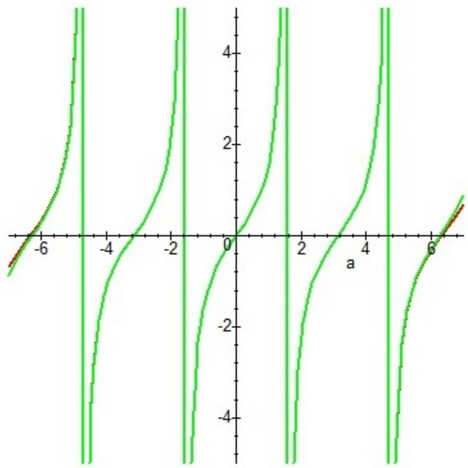
Here ‘a’ must not get too large for these approximations to hold.

We find that the n=4 approximation for the tangent is-

$$T(4,a)=\frac{[6930a^5-36a^7+2027025a-270270a^3]}{[a^8-630a^6+2027025+51975a^4-945945a^2]}$$

Plotting T(4,a) versus tan(a) produces the following result-

COMPARISON OF THE n=4 APPROXIMATION T(4,a) WITH TAN(a)



departture first noticed when |a|>6

The agreement is seen to be excellent in the range -6<a<6 clearly showing the infinities of tan (a) at a=(2n+1)π/2. For this n=4 case we find the approximations at 60 degrees to be-

$T(4,\pi/3)=1.7320508075687$

$C(4,\pi/3)=0.5000000000000$

$S(4,\pi/3)= 0.8660254037844$

All three of these trigonometric functions are accurate to 13 decimal places when a=π/3 and n=4.

CONCLUDING REMAKS:

We have shown that integrands involving the product of even Legendre polynomials P(2n,x) multiplied by a slowly varying function f(a,x) and integrated over the range 0<x<1 can produce excellent approximations for a function g(a) . For the technique to work it is necessary that the integral of P(2n,x) over 0<x<1 be zero which it is. Only the even Legendre polynomials from the extended family of oscillatory orthogonal polynomials obeys this condition. Tchebyshev and other polynomials fail to do so. The digit accuracy increases with increasing n and appropriate range for 'a' where g(a) is well approximated depends upon the form of the function f(x,a).The present technique is well suited for producing extensive math tables with a minimum of effort and also for supplying the values of g(a) to any specified order of accuracy for a given ‘a’.

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