THE FOUR RULES FOR LOGARITHMS

Logarithms were first discovered by the Scottish mathematician John Napier back in 1614. He showed that any number N can be rewritten as a base b taken to an exponent known as the logarithm of N. Mathematically one has the definition-

$$N=b^{\log_b(N)}$$

Here the base can be any number although most discussions in the literature confine themselves to just three. These are first the natural logarithms with b=e=2.71828…denoted as ln(N). Next one has the common logarithms where b=10 and the logarithm is denoted by log(N). and finally b=2 known as binary logarithms designated as $\log_2(N)$.

We wish here to quickly derive the four basic exponential laws for logarithms based on the above definition. Using common logarithms, we have-

$$x\cdot y=10^{\log(xy)}=10^{(\log(x)+\log(y)}$$

Looking at the exponent terms we get the first rule-

$$\log(xy)=\log(x)+\log(y)$$

This result is actually valid for any base b. Thus –

$$\log_2(32)=\log_2(8)+\log_2(4)$$

produces 5=3+2.

Next look at-

$$x/y=10^{\log(x)}/10^{\log(y)}=10^{(\log(x)-\log(y))}$$

Taking the exponential part, yields-

$$\log(x/y)=\log(x)-\log(y)$$

So log(100/10)=1=2-1

For the third rule we start with and use properties of exponentials. This produces- x^n=b^(log(x^n))=b^nlog(x). Thus-

$$\log(x^n)=n\log(x)$$

as the third law.

Finally using the above definition formula, we get-

$$x=b^{\log_b x}=a^{\log_a(x)}$$

Letting x=a and noting that $\log_a(a)=1$, produces $a=b^{\log_b(a)}$. Plugging this into the x equation produces the result-

$$\log_b(x)=\log_b(a)\,\log_a(x)$$

as our 4th rule. If we now take b=e and a=10, we get-

$$\ln(x)=\ln(10)\log(x) \quad \text{so that} \quad \ln(x)/\log(x)=2.30258\ldots$$

So if we know ln(2)=0.693147 we have at once that log(2)=0.301029

With the above four rules plus the original definition of a logarithm we can now evaluate an unlimited number of equations involving logarithms.

Let us start with-

$$\ln(x^3)=x$$

This can be written as-

$$\ln(x)/x=1/3$$

The substitution x=exp(u) then produces-

$$-u\exp(-u)=-1/3$$

Recalling that LambertW(-uexp(-u))=-u, we get-

$$-u=-\ln(x)=\text{LambertW}(-1/3)$$

So the final closed form solution becomes-

$$x=\exp(-\text{LambertW}(-1/3))= 1.857183860\ldots$$

Next consider the equation-

$$\ln(x)+\ln(x+2)=3$$

We can rewrite this as –

$$x(x+2)=\exp(3)$$

On solving this quadratic, one finds the two solutions-

$$x=-1\pm\text{sqrt}[1+\exp(3)]=-1\pm 4.591899054\ldots$$

Next we show how two numbers can be multiplied together by using logaritms. Consider N=57and M=78. Here we have-

$$NM=10^{\wedge}(1.755874856+1.892094603)=10^{\wedge}(3.647969459)$$

This may be written as-

$$NM=10(3+0.647969459)$$

Using a log table one finds that 10^mantissa=10^.647969=4.445.953. Multiplying this result by 10^3, we get the result-

$$NM=4445.953$$

Rounding things off we arrive at the answer-

$$NM=4446$$

This approach to finding the product of two numbers has now become obsolete in view of the existence of pocket calculators which find the value of NM directly without the need for first finding logarithms.

As a final evaluation consider a logarithmic equation involving two different bases. The equation we have in mind reads-

$$\log(x^{\wedge}3)=4+\ln(\text{sqrt}(x))$$

or the equivalent-

$$3\log(x)=4+(1/2)\ln(x).$$

But we know from the above earlier base change formula that ln(x)=ln(10)log(x). So we have the equation-

$$\log(x)=4/(3-(1/2)\ln(10))=2.1636738\ldots$$

Taking the exponent then yields the final answer-

$x=8.7030528\ldots$

U.H.Kurzweg
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