PI EVALUATION BY THE ARCHIMEDES METHOD, LUDOLPHINE NUMBER, AND SIMPLE EVALUATION BY ITERATION

Some 2300 years ago the famous mathematician and all around polymath Archimedes of Syracuse (287-212BC) introduced the first algorithm for determining the bounds on the irrational number π. His method was to determine the inner and outer circumference of a regular polygon with n vertexes relative to a unit circle . Based on modern notation he showed that-

$$n \sin(\pi/n) < \pi < n \tan(\pi/n)$$

Bounds for n=4 (square) through n=8(octagon) yield the values-

n	n sin(π/n)	n tan(π/n)
4	2.82842	4.00000
5	2.93892	3.63271
6	3.00000	3.46410
7	3.03718	3.37192
8	3.06146	3.31370

Although the above formulas will yield π as n goes to infinity, it does so very slowly. Even when n=10,000, one gets only a six digit accuracy for $\pi \approx 3.141592$ .The last person to use this approach for finding π to a large number of places was Ludolph van Ceulen(1540-1610). He actually spent twenty years as a professor at Leiden University in the Netherlands using the Archimedes method. He was able to accurately show the value of π out to 35 digits. His result was-

3. 14159 26535 89793 23846 26433 83279 50288

He was so proud of his result that he had the number engraved on his tombstone in Leiden. For many years this approximation to π was referred to in Germany as the Ludolphine Number in honor of its native son from Hildesheim. Poor Ludolph, little did he realize that within a century the introduction of calculus and the use of arctan formulas would make the calculation for much higher digit number using the Archimedes method obsolete. Today, with the use of supercomputers , one can readily find the irrational number π to about 10^14digits of accuracy using either AGM methods or iteration. On my home laptop ,using Maple, I can easily get the value of π accurate to ten thousand places in a split second.

Finally let us show how modern day iteration of the simple definition-

$$\pi=4\ \arctan(1)$$

can lead to any desired accuracy for π. We start with the two term expansion about x=a. It reads-

$$\arctan(x)=\arctan(a)+(x-a)/(1+a^2)$$

Next we set arctan(x)=x[n+1] and arctan(a)=x[n]. One also has a=tan(x[n]).

A little substitution yields the iteration equation-

$$x[n+1]=x[n]+\cos(x[n]^2])(1-\tan(x[n])$$

which produces-

$$x[n+1]=x[n]+\cos(x[n])^2-(1/2)\sin(2x[n]) \quad \text{with} \quad x[0]=1$$

We expect x[∞]= π/4=0.785398… . A simple one line program using Maple produces a list of iteration values. Looking at just 4(x[6]) produces-

3.1415926535897932384626433832795028841972217034369

The digits marked in red represent the Ludolphine Number here gotten in a split second.

U.H.Kurzweg
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Gainesville, Florida