WHAT ARE MAGIC SQUARES AND HOW ARE THEY CONSTRUCTED?

A magic square is any n x n array of numbers where each of the n^2 elements appears only once. Also the sum of the elements in each row, column, and diagonal have the same value. Such squares have been known since ancient times in both China and India and continue to draw the attention of professional and amateur mathematicians to the present day with the purpose of understanding any additional properties and their relation to Lie algebras. Historical interest in magic squares started a long time ago in China with the 3x3 square (LoShu, 650BC), followed later by the 4x4 squares of the hermitic philosopher Agrippa(1510AD) and artist Dürer(1514 AD), and the 8x8 and 16 x 16 squares discovered by the American statesmen , printer, and scientist Benjamin Franklin(1706-1790). The work by the mathematician Leonard Euler(1707-1783) on Latin squares and the recent interest in the number puzzle Sudoku by the public have also contributed heavily to interest in magic squares. We want here to look at some of the properties of these squares and to discuss the ways in which they may be constructed.

We start by considering an n x n magic square. This has a total of n rows and n columns. Adding together all n^2 elements, produces the total sum-

$$T = 1 + 2 + 3 + 4 + ... + n^2 = \frac{n^2(n^2+1)}{2}$$

Since there are n columns, we have that the sum in any row , column, or diagonal will be exactly-

$$S = \frac{n\ (n^2+1)}{2}$$

For a 3x3, 4x4, 5x5, 6x6, 7x7, 8x8, 9 x 9, and 10 x 10 magic squares the sum of the integers in any row, column, or diagonal will be 15, 34, 65,111, 175, 260, 369, and 505, respectively.

Consider first a 3x3 magic square which we represent by the square matrix-

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix}$$

It has a total of nine unknowns but only eight equations defining a magic square of this dimension. The eight equations are-

A+B+C=15, D+E+F=15, G+H+I=15, A+D+G-15, B+E+H=15, C+F+I=15,

A+E+I-15, and G+E+C=15

Eliminating seven of the unknowns, we arrive at the matrix-

$$\begin{vmatrix} A & B & 15-A-B \\ 20-2A-B & 5 & 2A+B-10 \\ A+B-5 & 10-B & 10-A \end{vmatrix}$$

One must now choose integer values of A and B in such a manner that all ten integers 1 through 9 appear only once in the matrix. Also we require that 14≥A+B≥6 to keep within the magic square construction rules. Solutions that work are [A,B]=[4,3] , [2,9], [6,7], and [8,1]. They produce the magic squares-

$$\begin{vmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{vmatrix} , \begin{vmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{vmatrix} , \begin{vmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{vmatrix} \quad and \quad \begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix}$$

These four are essentially the same square as simple successive ninety degree rotations show. It is interesting to note that this magic square is essentially the same as first given by the Chinese some twenty six hundred years ago. Notice that all integers from 1 through 9 appear in these squares only once as required by the definition. Also one notes that adding the same constant k to every element of a normal magic square produces another square which in the strict sense is no longer a magic square but does still have all of its rows, columns, and diagonals equal to the same constant 15+3k. On setting k=0, 9, 18, and 27 we find-

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{vmatrix} , \begin{vmatrix} 17 & 10 & 15 \\ 12 & 14 & 16 \\ 13 & 18 & 11 \end{vmatrix} , \begin{vmatrix} 26 & 19 & 24 \\ 21 & 23 & 25 \\ 22 & 27 & 20 \end{vmatrix} , \begin{vmatrix} 35 & 28 & 33 \\ 30 & 32 & 34 \\ 31 & 36 & 29 \end{vmatrix}$$

These four matrices contain all integers 1 through 36 and thus might form the basis for a 6 x 6 magic square. Let us try the arrangement-

$$\begin{vmatrix} 8 & 1 & 6 & 26 & 19 & 24 \\ 3 & 5 & 7 & 21 & 23 & 25 \\ 4 & 9 & 2 & 22 & 27 & 20 \\ 35 & 28 & 33 & 17 & 10 & 15 \\ 30 & 32 & 34 & 12 & 14 & 16 \\ 31 & 36 & 29 & 13 & 18 & 11 \end{vmatrix}$$

where the matrices of intermediate element size are placed at the right. Adding up the sum of the elements in each of the six rows, one sees that they all match the expected value of $6(3^2+1)/2=111$. However the first three rows each add up to 84 and the next three rows add to 138. So there is a net 27 deficit or surplus for each of the six rows. We can adjust this by exchanging numbers in a given row in order not to disturb the column sums. There are many ways to do this. The simplest is to use the following three interchanges-

$$8 \leftrightarrow 35 \qquad 5 \leftrightarrow 32 \qquad 4 \leftrightarrow 31$$

which makes all the rows plus the two diagonals equal to 111. We thus have the valid 6 x 6 magic square-

$$\begin{vmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{vmatrix}$$

One can also extend the above derivation for a 6 x 6 magic square to the higher values n=12, 24, 48, ..$6\cdot 2^k$.

It should be pointed out that there are many other versions of such 6 x 6 magic squares For instance the one given in 1510 by Heinrich Cornelius Agrippa looks completely different yet is as legitimate as the one derived above. Agrippa lists the following seven magic squares-

FIRST SEVEN MAGIC SQUARES

4	9	2
3	5	7
8	1	6

3 x 3

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

4 x 4

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

5 x 5

6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

6 x 6

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

7 x 7

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

8 x 8

37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

9 x 9

At that time in history philosophers indulged heavily in hermiticism and numerology and Agrippa actually claimed a relation between the seven squares shown above and the planets, sun, and moon. One can make several observations regarding the odd n magic squares shown in this last figure. First of all, a single central element exists only for odd n squares. The values are 5-13-25-41 for squares with n=3, 5, 7, and 9, respectively. This means one should find this central element to have the odd integer value-

$$N = \frac{(n^2 + 1)}{2}$$

regardless of the odd number value. Also we observe that for odd n the outer two rows and two columns alternate in even or odd character . In addition the element just below the central element equals 1 and that just above n^2 in Agrippa's odd n magic squares.

Let us next look at the 4x4 magic square of Dürer as it appears in his 1514 engraving Melencholia . Here we have a total of 16 elements where each row, column, or diagonal adds up to S=34. It has the explicit form-

$$\begin{vmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{vmatrix}$$

You will note that it looks close to that given by Agrippa with only a slight change in the number pattern. Both squares have the interesting property that there are four sub-matrices whose elements also sum to 34. The upper right sub-matrix for Dürer's square has the element sum of 2+13+11+8=34. Treating this as a Sudoku problem, we have a total of 14 equations (4 rows, 4 columns, 2 diagonals, and 4 sub-matrices) for a total of 16 unknown elements. In generic form we have-

$$\begin{vmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ O & P & Q & R \end{vmatrix}$$

The governing equations are –

A+B+C+D=34, E+F+G+H=34, I+J+K+L=34, O+P+Q+R=34, A+E+I+O=34, B+F+J+P=34, C+G+K+Q=34, D+H+L+R=34, A+F+K+R=34, D+G+J+O=34, A+B+E+F=34, C+D+H+G=34, K+L+R+Q=34, and I+J+P+O=34

As we did with the earlier 3 x 3 case, we can eliminate all unknowns except three(say, A, B, and C). With an appropriate choice for these remaining unknowns one recovers Duerer's magic square. The particular choice will be [A,B,C]=[16,3,2].

Having one of the forms of a 4 x 4 magic square, we can proceed to work out larger squares of the form=$4\cdot 2^k$. Benjamin Franklin did this for n=8 and n=16 squares. Here is his 8 x 8 square-

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Note the elements in all rows and columns add up to 4(64+1)=260 as expected, but his two diagonals add up to 292 and 228 which means they depart from 260 by ±32. This shows that the Franklin square is not a true magic square. A true 8 x 8 magic square is that given by Agrippa as found above or our own version-

$$\begin{vmatrix} 64 & 51 & 2 & 13 & 32 & 19 & 34 & 45 \\ 53 & 58 & 11 & 8 & 21 & 26 & 43 & 40 \\ 57 & 54 & 7 & 12 & 25 & 22 & 39 & 44 \\ 52 & 63 & 14 & 1 & 20 & 31 & 46 & 33 \\ 16 & 3 & 50 & 61 & 48 & 35 & 18 & 29 \\ 5 & 10 & 59 & 56 & 37 & 42 & 27 & 24 \\ 9 & 6 & 55 & 60 & 41 & 38 & 23 & 28 \\ 4 & 15 & 62 & 49 & 36 & 47 & 30 & 17 \end{vmatrix}$$

where the sum of the elements in any row, column, <u>and</u> diagonals equals precisely 260. I leave it to the reader to retrace the steps I used to derive this 8 x 8 magic square. (Hint- we used Dürer's magic square as a starting point).

Finally I leave you with a couple of 12 x 12 magic squares-

114	140	3	34	35	1	78	104	75	106	107	73
7	119	135	28	8	30	79	83	99	100	80	102
19	14	124	123	23	24	91	86	88	87	95	96
18	20	130	129	17	13	90	92	94	93	89	85
25	137	118	9	26	12	97	101	82	81	98	84
144	113	33	4	2	31	108	77	105	76	74	103
6	32	111	142	143	109	42	68	39	70	71	37
115	11	27	136	116	138	43	47	63	64	44	66
127	122	16	15	131	132	55	50	52	51	59	60
126	128	22	21	125	121	54	56	58	57	53	49
133	29	10	117	134	120	61	65	46	45	62	48
36	5	141	112	110	139	72	41	69	40	38	67

and-

143	109	6	26	19	24	107	73	78	98	91	96
3	140	115	21	23	25	75	104	79	93	95	97
31	9	110	130	27	20	103	81	74	94	99	92
8	28	141	125	10	15	80	100	105	89	82	87
30	113	142	12	14	16	102	77	106	84	86	88
112	144	29	13	18	11	76	108	101	85	90	83
35	1	114	134	127	132	71	37	42	62	55	60
111	32	7	129	131	133	39	68	43	57	59	61
139	117	2	22	135	128	67	45	38	58	63	56
116	136	33	17	118	123	44	64	69	53	46	51
138	5	34	120	122	124	66	41	70	48	50	52
4	36	137	121	126	119	40	72	65	49	54	47

You will notice that every row, column, and diagonal equals the same value of 12(144+1)/2=870. See if you can figure out how I was able to come up with these results starting with information given earlier on this page.