CONSTRUCTING MODIFIED PASCAL TRIANGLES

It is well known since the time of Newton that one can expand (1+x)^n as-

$$(1+x)^{\mathbf{n}} = 1 + \mathrm{nx} + \left[\frac{\mathbf{n(n-1)}}{\mathbf{2!}}\right]\mathbf{x^2} + \left[\frac{\boldsymbol{n}(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2})}{\mathbf{3!}}\right]\boldsymbol{x}^{\mathbf{3}} + \cdots = \sum_{\boldsymbol{m}=\mathbf{0}}^{\boldsymbol{n}}\left[\frac{\boldsymbol{n}!}{\boldsymbol{m}!(\boldsymbol{n}-\boldsymbol{m})!}\right]x^\wedge n$$

Here the term in the square bracket is the binomial coefficient C[n,m]. This coefficient can also be written in form of the standard Pascal Triangle-

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

with the rows starting with n=0. Also you will notice that C[n,m] equals the sum of the two elements lying directly above it in row n-1 and 0≤m≤n. The sum of the elements in each row equals 2^n. Reading off of the triangle for n=4 we get the identity-

(1+x)^4=1+4x+6x^2+4x^3+x^4

As first shown by us some five years ago (see https://www2.mae.ufl.edu/~uhk/MORE-PASCAL)) one can replace (1+x)^n by the function $1/[1-\exp(-x)]^n$ to generate a brand new type of Pascal Triangle given as-

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

Here as n gets large the elements along a fixed row n approach the shape of a Gaussian. This fact allows one to use this triangle to get a good approximation for n! at large n.

Lets begin with finding some additional Pascal like triangles. Many of these can be generated without having a guide function like in the above cases. One such array is-

1									
1	1								
1	3	1							
1	5	5	1						
1	7	13	7	I					
1	9	25	25	9	1				
1	11	41	63	41	11	1			
1	13	61	129	129	61	13	1		
1	15	85	231	321	231	85	15	1	
1	17	113	377	681	681	377	113	17	1

Here we see that-

$$C[n,m]=C[n-1,m-1]+C[n-1,m]+C[n-2,m]$$

For the special case of C[6,3]=C[5,2]+C[5,3]+C[4,2] we have 63=25+25+13. Adding up the elements in a given row n leads to S(0)=1, S(1)=2, S(2)=5, S(3)=12, S(4)=29, S(5)=70.Unfortunatekly this produces no obvious general term S(n).

Another Pascal like triangle is-

1														
2	2													
3	4	3												
4	6	6	4											
5	8	9	8	5										
6	10	12	12	10	6									
7	12	15	16	15	12	7								
8	14	18	20	20	18	14	8							
9	16	21	24	25	24	21	16	9						
10	18	24	28	30	30	28	24	18	10					
11	20	27	32	35	36	35	32	27	20	11				
12	22	30	36	40	42	42	40	36	30	22	12			
13	24	33	40	45	48	49	48	45	40	33	24	13		
14	26	36	44	50	54	56	56	54	50	44	36	26	14	
15	28	39	48	55	60	63	64	63	60	55	48	39	28	15

The individual elements are found (without need for a guide function) by noting that –

$$C[n,m]=C[n-1,m-1]+C[n-1,m]-C[n-2,m-1]+1$$

It is easy to find the element C[n,m] by noting the Cs are given by placing them at the four corners of a rhombus with the sum of the two vertical Cs greater by one to the sum of the two horizontal Cs. So C[14.5]=60 follows from-

45
50 54 or 50+54+1=45+60
60

Here the sum for the nth row equals-

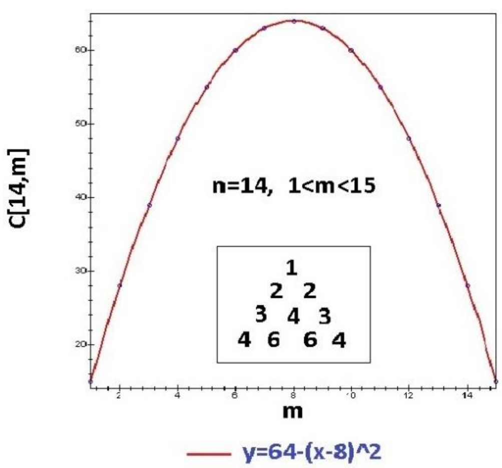
$$S(n)=[(n+1)(n+2)(n+3)]/6 =(n+3)!/(n!3!)$$

One will get a parabolic and not a Gaussian shape when plotting C[n,m] over the range 0<m<n at large n. As an example for n=14 we have the elements-

15 28 39 48 55 60 63 64 63 60 55 48 39 28 15

which yields the structure-

PARABOLIC ELEMENT STRUCTURE FOR THE MODIFIED PASCAL TRIANGLE SHOWN



As a final Pascal like triangle consider-

1
1 1
1 6 1
1 16 16 1
1 36 76 36 1
1 76 256 256 76 1

Here C[n,m] is obtainrd by summing the nearest three Cns above it and multiplying by two. So, for example, 36=2*I1+16+1] and 256=2*[36+76+16].

U.H.Kurzweg
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Gainesville, Florida