MOMENTS OF INERTIA

The moment of inertia I about a given axis is defined as-

$$I = \int r^2 dm = k^2 m$$

where r is the perpendicular distance from the axis to an increment of mass dm, k is the radius of gyration, and m is the total mass of the body. Let us consider first how one obtains I and k for a disc of radius r=a, thickness t and constant density ρ_0. As the rotation axis consider the z axis and let the disc lie parallel to the x-y plane. Using cylindrical coordinates one has-

$$I_{zz} = \rho_0 \int_0^t dz \int_{r=0}^{a} r^3 dr \int_{\theta=0}^{2\pi} d\theta = \frac{m}{2} a^2 \;\; with \; m = \rho_0 \pi a^2 t \; and \; k = \frac{a}{\sqrt{2}}$$

Next consider a thin flat plate of mass m and sides 2a and 2b rotated about a normal axis through its center. This time we use cartesian coordinates to get-

$$\frac{I_{zz}}{\rho_0 t} = 4 \int_{x=0}^{a} dx \int_{y=0}^{b} (x^2 + y^2) dy = \frac{4}{3} ab(a^2 + b^2)$$

The Moment of Inertia about the x axis through the center of this plate produces-

$$\frac{I_{xx}}{\rho_0 t} = 4 \int_{x=0}^{a} x^2 dx \int_{y=0}^{b} dy = \frac{4}{3} ba^3$$

and I_{yy} about the y axis through the center produces-

$$\frac{I_{yy}}{\rho_0 t} = 4 \int_{x=0}^{a} dx \int_{y=0}^{b} y^2 dy = \frac{4}{3} ab^3$$

Notice that for this plate, as for any lamina, one has the perpendicular axis theorem-

$$I_{zz} = I_{xx} + I_{yy}$$

To get the Moment of Inertia about an axis parallel to the z axis about a corner of this thin plate at x=a and y=b, one has-

$$I_{z'z'}=\rho_0 t\int_{x=-a}^{a}dx\int_{y=-b}^{b}[(x-a)^2+(y-b)^2]dy=(\rho_0 t)\frac{16}{3}ab(a^2+b^2)$$

Note that this result is a equivalent to-

$$I_{z'z'}=I_{zz}+md^2$$

where d=sqrt(a^2+b^2) is the distance between the two parallel axes. This last result is just the parallel axis theorem which is valid for all bodies not just laminas.

Consider next a more complicated problem of a I about any diameter of a constant density(ρ=1) sphere of radius r=a. This problem is easiest to solve by stacking up a series of discs of radius r=sqrt(a^2-z^2) with the dI of each disc being dI=$\rho_0\pi r^4$dz/2. This leads to-

$$I_{zz}=\frac{\rho_0\pi}{2}\int_{z=-a}^{a}(a^2-z^2)^2dz=\frac{8}{15}\rho_0\pi a^5 \text{ which yields } I_{zz}=\frac{2}{5}ma^2$$

For a cylinder of radius r=a and height z=H the moment of inertia about an its axis through its axis is found by stacking up the incremental moments of disc of radius r=a and thickness dz. This produces-

$$I_{zz}=\frac{\rho_0\pi a^4 H}{2}=\frac{1}{2}ma^2$$

We see that the radius of gyration of a sphere is smaller than that of a cylinder. Hence if you are ever asked which body of the same mass m will roll faster down an inclined plane you can answer at once that it will be the sphere since its angular acceleration α will be higher.

When dealing with bodies of constant density ρ_0 with cavities in them one first calculates the I_{zz} of the body without cavities and then subtracts the moment of inertia of the cavities by replacing them with material of the same density as that of the cavity free body. Thus a spherical shell of outer radius r=a and inner radius r=b has the moment of inertia-

$$I_{zz}=\frac{8}{15}\rho_0\pi\{a^5-b^5)$$

Finally let us look at a more challenging problem, namely, that of the momemt of inertia of a uniform density cube of sides '2a' about a diagonal axis AB passing through opposite corners at(a,a,a) and(-a,-a,-a). The unit length vector along this axis is λ=(i+j+k)/sqrt(3). If we know go to the definition of I_{AB} -

$$I_{AB} = \rho_0 \iiint p^2 dxdydz = \rho_0 \iiint |\lambda \times (ix + jy + kz)|^2 \; dxdydz$$

where p is the perpendicular distance from the axis AB to the increment of mass $\rho_0$dxdydz. Here ix+jy+kz is the vector coming from the origin to the point mass dm. On multiplyning things out and noting the symmetry of the body about the origin at the cube center, we are left with-

$$I_{AB} = \frac{2}{3}\rho_0 \iiint (x^2 + y^2 + z^2) dxdydz = \frac{16}{3}\rho_0 a^5 = \frac{2}{3} ma^2$$

It is interesting in this case that the principal moments of inertia are given by-

$$I_{zz} = \rho_0 \iiint (x^2 + y^2) dxdydz = \frac{16}{3}\rho_0 a^5 = I_{xx} + I_{zz}$$

and thus have the same value as that about the diagonal axis AB. Note that products of inertia will enter these calculations when the body is asymmetric.