MORE ABOUT FACTORIALS

Back in 2013 we discussed in detail on this web page some of the properties of n! and the related gamma function Γ(n+1). We want here to look at some additional properties of n!. We begin the discussion with the most elementary description of a factorial. It is-

$$1 = 1 = 1!$$

$$1 \cdot 2 = 2 = 2!$$

$$1 \cdot 2 \cdot 3 = 6 = 3!$$

$$1 \cdot 2 \cdot 3 \cdot 4 = 24 = 4!$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120 = 5!$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720 = 6!$$

One sees at once from this table that the point function n! satisfies the generating formula-

$$(n+1)! = (n+1)n!$$

According to this formula we have 0!=1 and 7!=720·7=5040 , 8!=5040·8=40320 and 9!=40320·9=362880. For negative even integers we have [-(2n)]!=-∞ and for negative odd integers we get [-(2n+1)}!=+∞. By definition one also has that-

$$[(n+1)(n+2)...(m-1)(m)] = \frac{m!}{n!}$$

Thus 7·8·9=9!/6!=504.

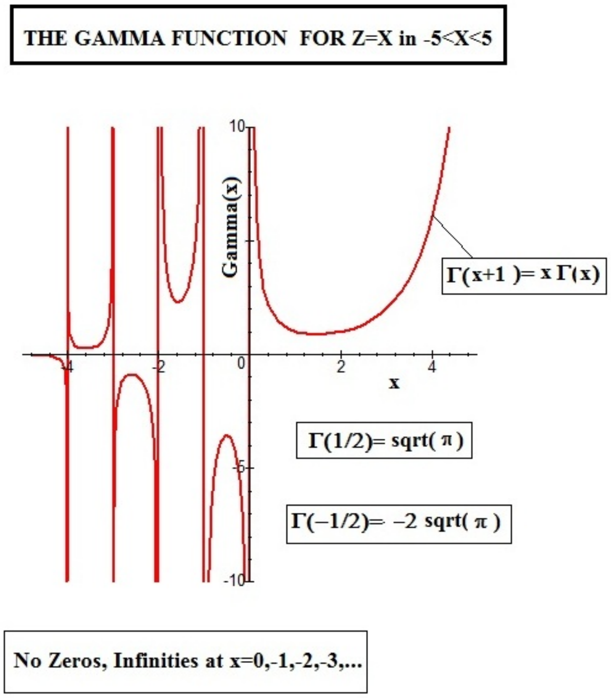
To fill in the spaces between integers it is necessary to introduce the integral identity-

$$\int_{t=0}^{\infty} t^n \exp(-t)dt = Laplace\{t^n \ with \ s = 1\} = \Gamma(n+1)$$

Here Γ(x+1) is the Gamma Function. It is continuous function except for poles at negative integers x. One also has the important identity-

$$\Gamma(n+1) = n!$$

A plot of the Γ(x) function follows-



The factorials n! lie at integer values of x along this curve. Thus Γ(2)=1!, Γ(3)=2!, andΓ(4)=3! The generating function, as readily deducible from the earlier factorial formula, is-

$$\Gamma(x+1) = x\Gamma(x)$$

The Gamma Function at x=1/2 follows from the definition-

$$\Gamma(\frac{1}{2}) = \int_{t=0}^{\infty} \frac{\exp(-t)}{\sqrt{t}} dt = 2\int_{u=0}^{\infty} \exp(-u^2) du = \sqrt{\pi}$$

From this last result one can generate all other half-integer Gamma Functions. So we have, for instance,-

Γ(-1/2)= -2sqrt(π) and Γ(3/2)= sqrt(π)/2

Typical math tables for the Gamma Function only give Γ(x) in the range 1<x<2 since all values outside this range can readily be deduced using the above generating formula. The minimum value of Γ(x) for positive x is 0.8856 at x=1.46.

Getting back to n!, we can write=

$8! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8 = (1\cdot3\cdot5\cdot7)2^4 4!$

On generalizing this to n, we get the identity-

$$(2n)! = n!2^n \prod_{k=1}^{n} (2k-1)$$

This product term is often encountered in infinite series expansions of certain functions. Take for example the Taylor series expansion-

$$\frac{1}{\sqrt{1-x}} = 1 + (1/2)x + (3/8)x^2 + (5/16)x^3 + (35/128)x^4 + (63/256)x^5 + ...$$

Manipulating the coefficients, we see that 3/8= (1·3)/(2^2·2!), 5/16=(1·3·5)/(2^3·6) and 35/12=(1·3·5·7)/(2^4·24). Thus we have the compact infinite series expansion-

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{k=1}^{\infty} \frac{[1\cdot 3\cdot 5\cdot ...\cdot(2k-1)]}{2^k \cdot k!} x^k = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} x^k \qquad |x|<1$$

I remember over fifty years ago, while a freshman in my introductory calculus course, that I was somewhat puzzled how such compact coefficients came about. Their derivation became clear to me in subsequent years after learning how one manipulates factorials.

We point out that sometimes one talks about the double factorials of an odd number. Such double factorials are defined as-

$$\text{(2n-1)!!=1·3·5·…·(2n-1} = \prod_{k=1}^{n} (2k-1)$$

Since we know the value of the product Π from above, we can also write-

$$\text{(2n-1)!!} = \frac{(2n)!}{n!2^n}$$

So we get 7!!=105 and 9!!=945. Since n! is only defined as products of whole integers, double primes for even ns are not possible since this would involve half integer values of n.

A more general expansion involving factorials is the Binomial Expansion-

$$(x+y)^n = x^n + \frac{nx^{n-1}y}{1!} + \frac{n(n-1)x^{n-2}y^2}{2!} + ... = \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} x^{n-k} y^k$$

The coefficient in this expansion is the Binomial Coefficient usually designated by C_k^n. We prefe, however, the designation C[n,k] as it is easier to type out. This coefficient also appears in the well known Pascal Triangle-

```
              1
           1     1
        1     2     1
      1    3     3     1
    1   4     6     4    1
  1   5   10   10    5     1
```

Note here that each row n ,starting with n=0, adds up to 2^n and that a coefficient in the nth row C[n,k] equals the sum of the two coefficients directly above it. That is-

C[n,k]=C[n-1,k-1]+C[n-1,k]

The columns of constant k are diagonals in the Pascal triangle. In pre-calculus days mathematicians would use the Pascal Triangle to calculate the Binomial Coefficients. So to get C[5,2] they just added up 4 and 6 to get 10. Several years ago we discovered a modified version of the Pascal Triangle looking like this-

```
                                                sum
                  1                              1!
              1       1                          2!
          1       4       1                      3!
      1      11      11       1                  4!
   1    26      66      26       1               5!
1   57     302     302      57       1           6!
```

Although no direct applications of this triangle array have been found as yet, it is clear that the triangle has the properties that the sum of the elements in any row add up to the factorial of that row and there is a symmetry about a vertical line passing through 1-4-66. After some effort we were able to express the D[n,m] coefficients present in this modified triangle by the formula-

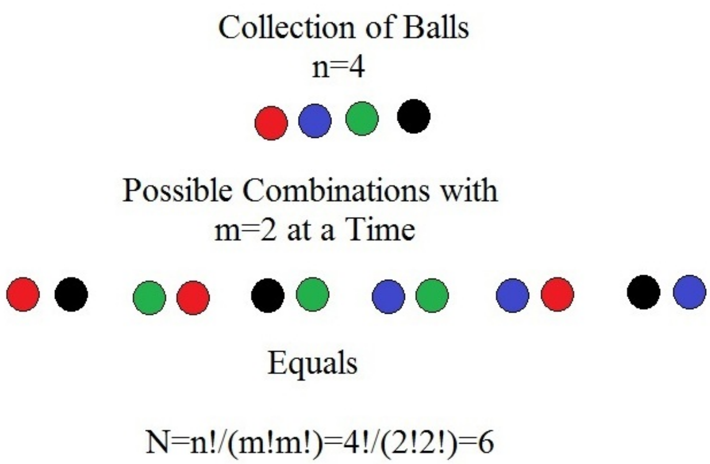
$$D[n,m]=\sum_{k=1}^{m}\frac{(-1)^{k-1}(n+1)!(m+1-k)^{n}}{(k-1)!(n+2-k)!}$$

heavily involving factorials. We have D[4,2]=11 , D[5,3]=56 and D[6,3]=302.

The Binomial Coefficient also arises in the area of statistics. One of the fundamental rules there is that if one has n objects and asks how many unique different combinations N of m objects at a time can be selected from these n objects, the answer is –

$$N = C[n,m] = \frac{n!}{m!(n-m)!}$$

Let us demonstrate this law by considering all the possible unique combination of four different color balls (n=4) picked two (m=2) at a time. One gets the following picture-



It clearly shows six unique combinations as predicted by the Binomial Coefficient C[4,2]=6.

Another famous formula expressible in factorials is the Wallis (1616-1703) Formula for π. It reads-

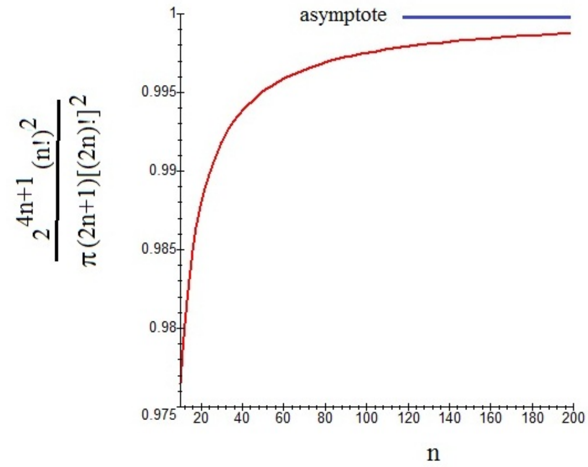
$$\frac{\pi}{2} = \frac{\lim}{n \to \infty}\{\frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot (2n)^2}{[1\cdot 1\cdot 3\cdot 3\cdot \ldots \cdot (2n-1)^2](2n+1)}\}$$

A bit of manipulation, using the results found above, allows us to recast his formula into the compact form-

$$\frac{\pi}{2} = \frac{\lim}{n \to \infty}\left\{\frac{2^{4n}(n!)^4}{(2n+1)[(2n)!]^2}\right\}$$

In this form one can get a feeling of what value the quotient on the right converges to as n is increased. Take n=10. It yields 1.533 while n=100 yields 1.566. The exact value to which this rather slowly converging series converges is π/2=1.57079.. . Its approach toward this limit is shown in the following graph-

CONVERGENCE OF THE WALLIS PRODUCT



An n a little over twenty terms(n=20) already shows a 99% accuracy.

An asymptotic value for n! as n gets large can be derived from the definition-

$$n! = \int_{t=0}^{\infty} \exp[-t + n\ln(t)]dt$$

The exponent here has a maximum at n=t and the integrand resembles a Gaussian with a peak value of n^nexp(-n). So approximating the integrand about t=n as a Gaussian by integrating things from minus to plus infinity, one finds the so-called Stirling (1692-1770) Approximation -

$$n! \approx \sqrt{2\pi n}\, n^n \exp(-n)$$

This formula was actually first found by Abraham deMoivre (1667-1754). An improvement , derivable via the method of steepest descent, multiplies this result by the factor {1+1/(12n)+1/(288n^2)-…}. This means the error in the above approximation is about 0.8% for n=10. The exact value for 10! is 3,265,920 while the above approximation yields the slightly lower value of 3,598,696. That is, the error is 0.829% , very close to the estimate using the first improvement. The above approximation has found a wide range of applications especially in statistical physics were n represents something like the number of molecules present in one cubic centimeter of gas at STP.

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