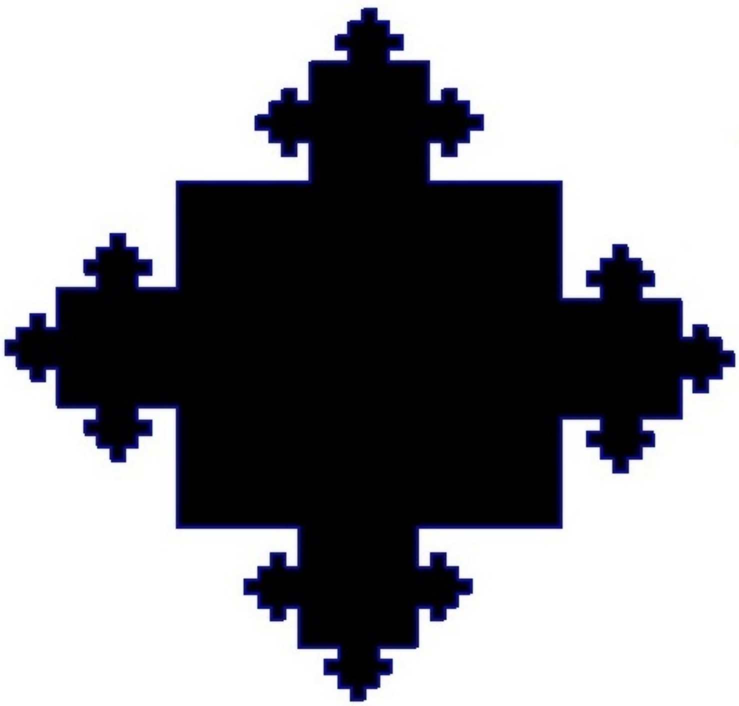
# PROPERTIES OF FRACTAL SQUARES

Several years ago while studying fractals, including the Koch Fractal Curve and the Sierapinski Triangle, we found an interesting new figure based on a zeroth generation square. We termed the higher generation fractal arising from this square the Black Snowflake. We wish here to look more at the properties of this snowflake and also another related fractal.

Our starting point is a figure of the four-fold symmetric Black Snowflake shown-



It is constructed by use of the concept of generations. Here the zeroth generation is a square of area $A_0= L^2$. Attached to this square are four first generation squares of area $A_1=L^2/9$ each. Next one forms the second generation by attaching 36 even smaller squares of area $A_2=L^2/81$ each to the first generation. Note that, unlike for a Koch Curve, the nth generation may only touch the (n-1) generation and never the (n-2), (n-3), etc generations. Continuing the addition of generations out to infinity, we get a total finite area of-

$$A = L^2\left\{1+\frac{4}{9}+\frac{12}{81}+\frac{36}{729}+...\right\} = L^2\left\{1+\left(\frac{2}{3}\right)^2[1+\frac{1}{3}+\frac{1}{3^2}+...]\right\} = \frac{5}{3}L^2$$

by use of the geometric series. We note that the distance from the center of the zeroth generation out to the infinite generation is just-

$$\Delta = L\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+...\right\}=1L$$

This means that if we enclose the snowflake in the smallest possible square, the square will have an area of $2L^2$. So that the ratio R between Snowflake area and to the circumscribing square area, rotated by 45deg, will be-

$$R=\frac{5}{6}=83.333\ percent$$

In terms of a Hausdorff dimension, the Black Snowflake has the dimension-

$$d=\frac{\ln(5)}{\ln(3)}=1.46497...$$

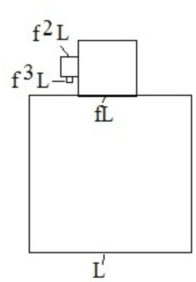
We can also calculate the perimeter of the snowflake by noting that the zeroth generation has the perimeter-

$$P_0=L\{4-(4/3)\}=8L/3$$

since part of its perimeter is blocked by the next generation. The contributions to the perimeter of all subsequent generations is also found to be 8L/3 for each. Thus the perimeter of the entire Black Snowflake, going out to the infinite generation, is infinite. Such behavior is to be expected when dealing with fractals having non-integer dimensions.

Since a condition for the Black Snowflake is that only the (n+1) and (n-1) may touch the nth generation, it is clear that there must be a restriction on the side length ratio of the nth generation square. To find what this restriction is, we look at the following diagram-

RESTRICTION ON f TO PREVENT OVERLAP




$$fL/2>f^2L/2+f^3L(1+f+f^2+f^3+..)$$

We see that each generation is represented by squares of side-length $f^n L$. So looking at the part going down from the $f^2 L$ square, we must have-

$$\frac{fL}{2} > \frac{f^2 L}{2} + f^3 L + f^4 L + f^5 L +,,,$$

Cancelling out the fL/2 term and making use of the geometric series since f<1, we find-

$$1 > 2f + f^2$$

This solves as –

$$f < \sqrt{2} - 1 = 0.41421356...$$

The Black Snowflake which has f=1/3 falls into the non-overlap case, but taking f=1/2 would produce a definite overlap after the third generation.

To see how many generations it would take to produce an overlap when f>sqrt(2)-1, we must have –

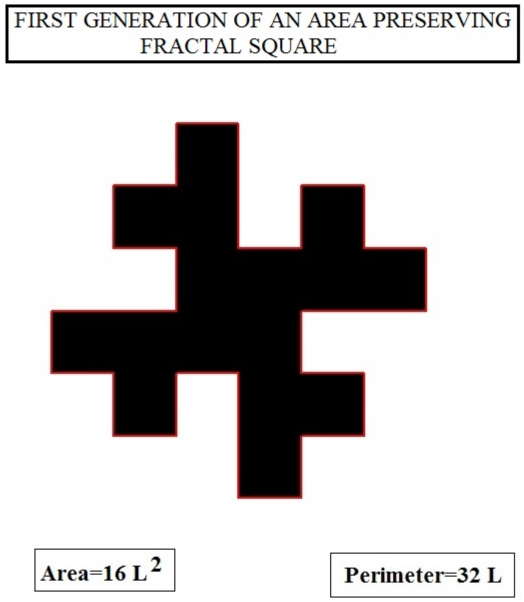
$$\frac{f}{2} - \frac{f^2}{2} = f^3 + f^4 + ... + f^n$$

After application of the finite geometric series, this is equivalent to saying –

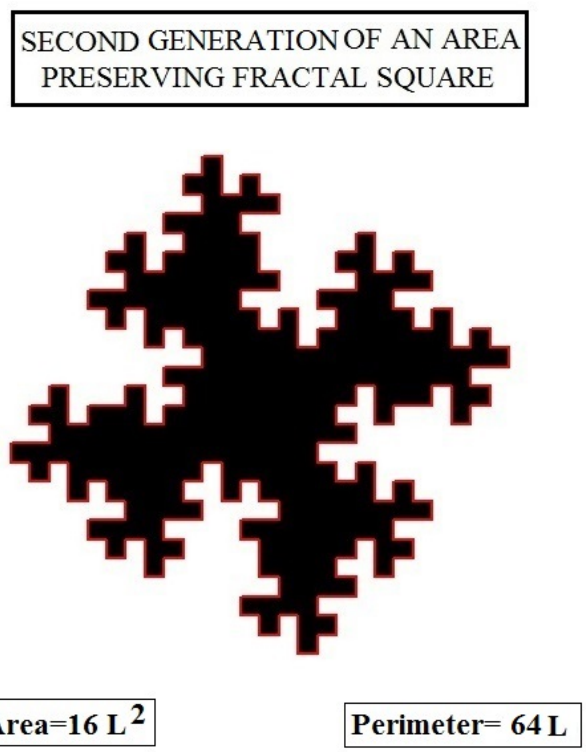
$$2f^n = f^2 + 2f - 1$$

for the beginning of overlap. Thus when f=1/2 overlap begins with the third generation(n=3).

One can produce many variations of the Black Snowflake fractal. One of the most interesting new configurations which we have just found starts with a square of side-length 4L for the zeroth generation. We break the sides into four equal length segments of L each and draw on each of the sides an outward square followed by an inward square of area $L^2$ representing the first generation. Since there is no change in effective area, the total area of the first generation remains at A=16 $L^{2.}$. However, the perimeter increases by a factor of two to 32 L. The first generation looks like this-



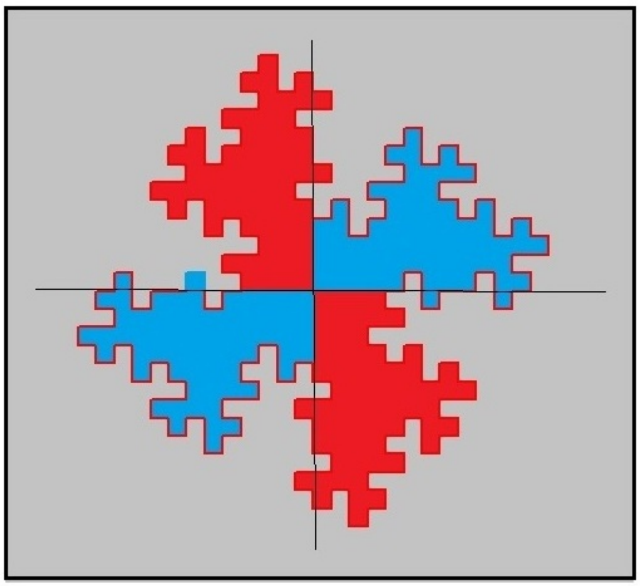

Following the same procedure for the second generation, we find the intricate fractal pattern –




It took some effort to draw this second generation following the same double pulse procedure. When I first saw this pattern it reminded me of a lion, a dancing clown, or a swastika. What struck me on first seeing the figure was that its area

remains the same as the original 4Lx4L square while the perimeter has increased by a factor of four. Clearly the ratio of perimeter to area will continue to increase with ever higher generations. This fact is something to keep in mind for those individuals involved with convective heat transfer or the design of improved air and water filters.
The four fold symmetry of the fractal square is shown more clearly in the following graph for the second generation-

FOUR FOLD SYMMETRY OF THE FRACTAL SOUARE



**There are 64 squares of area $L^2/16$ in each quadrant yielding a total area of $16L^2$**

We see that each quadrant contains 64 squares so that the total area equals-

$$64x4xL^2/16=16L^2$$

and so remains unchanged from earlier generations.

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