# PROPERTIES OF N-SIDED REGULAR POLYGONS

When students are first exposed to regular polygons in middle school, they learn their properties by looking at individual examples such as the equilateral triangles(n=3), squares(n=4), and hexagons(n=6). A generalization is usually not given, although it would be straight forward to do so with just a min imum of trigonometry and algebra. It also would help students by showing how one obtains generalization in mathematics. We show here how to carry out such a generalization for regular polynomials of side length s. Our starting point is the following schematic of an n sided polygon-

SCHEMATIC OF AN N-SIDED REGULAR POLYGON SHOWING THE BASE TRIANGLE USED IN ITS CONSTRUCTION
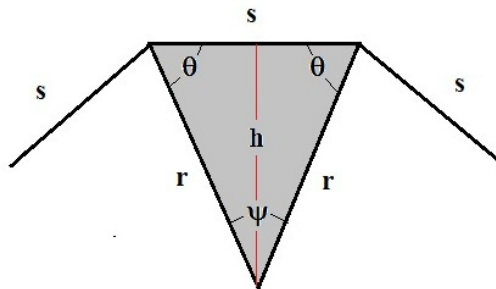
Polygon Area= $n(s/2)^2 \tan[(1-2/n)(\pi/2)]$

Polygon Perimeter=ns

Angle Relation $\psi = \pi - 2\theta = 2\pi/n$

Triangle Height $h=(s/2)\tan(\theta)$

We see from the figure that any regular n sided polygon can be constructed by looking at n isosceles triangles whose base angles are $\theta=(1-2/n)(\pi/2)$ since the vertex angle of the triangle is just $\psi=2\pi/n$, when expressed in radians. The area of the grey triangle in the above figure is-

$$A_{Tr}=sh/2=(s/2)^2\tan(\theta)=(s/2)^2\tan[(1-2/n)(\pi/2)]$$

so that the total area of any n sided regular convex polygon will be $nA_{Tr,}$, with s again being the side-length. With this generalized form we can construct the following table for some of the better known regular polygons-

Name	Number of sides, n	Base Angle, $\theta=(\pi/2)(1-2/n)$	Non-Dimensional Area, $4nA_{Tr}/s^2=\tan(\theta)$
Triangle	3	$\pi/6=30°$	1/sqrt(3)

Square	4	$\pi/4=45°$	1
Pentagon	5	$3\pi/10=54°$	sqrt(15+20φ)
Hexagon	6	$\pi/3=60°$	sqrt(3)
Octagon	8	$3\pi/8=67.5°$	1+sqrt(2)
Decagon	10	$2\pi/5=72°$	10sqrt(3+4φ)
Dodecagon	12	$5\pi/12=75°$	144[2+sqrt(3)]
Icosagon	20	$9\pi/20=81°$	20[2φ+sqrt(3+4φ)]

Here φ=[1+sqrt(5)]/2=1.618033989… is the well known Golden Ratio. It is seen that the base angle heads toward π/2 as n goes to infinity. Also the area of the polygon approaches πr^2 , where r is the radius of a circle centered in the polygon and passing through each of the polygon vertices. From the above diagram we have in addition that r=s/2cos(θ), so that for a regular convex polygon of an infinite number of sides the area is-

$$A = \frac{\lim}{n \to \infty}\left\{2h^2 \sin(\frac{2\pi}{n})\right\} = \pi r^2 \quad because \quad h \to r \ as \ n \to \infty$$

It is this limit which first allowed Archimedes to obtain good approximations for π.

Note that in terms of the base angle θ, the area of any n sided regular polygon will be-

$$A=n(s/2)^2\tan(\theta)=n(s/2)^2\tan[(\pi/2)(1-2/n)]$$

For each of the cases given in the above table, it is possible to construct the polygon using just a straight edge and a compass. Nineteen year old Karl Gauss showed in 1796 that one can always construct regular polygons if n satisfies the equality-

$$n = 2^m \{f_1 \cdot f_2 \cdot f_3 \cdot ....\}$$

 where m is a positive integer and f_n are any of the five known Fermat Primes-

$$f_k = 2^{2^n} + 1 = 3, 5, 17, 257, and \ 65537$$

Note that starting with $f_3=17$, all Fermat primes are of the form 8n+1 as deduced from our earlier discussion on the Integral Spiral. Thus the polygons which can be constructed by just straight edge and compass are in ascending order-

n=3, 4, 5, 6, 8, 10,12, 15, 17, 20 ,24, etc

Gauss's fame lies partially on the fact that he predicted the 17 sided heptadecagon could be constructed by using just a straight edge and compass. Note that 17 is the third Fermat Number. For this heptadecagon we have-

$$\theta=15\pi/34\,rad=79.4117647..^\circ \quad \text{and} \quad A=17(s/2)^2\tan(\theta)=17(s^2/4)\tan(15\pi/34)$$

Note also that one should be able to construct a 60 sided regular polygon with straight edge and straight edge since $60=2^2\cdot3\cdot5$. Such a polygon would be referred to as a hexacontagon.

Let us next demonstrate the procedure for constructing a pentagon(n=5) using only a straight edge and ruler. The procedure is shown in the following figure-

STRAIGHT EDGE AND COMPASS CONSTRUCTION OF A PENTAGON
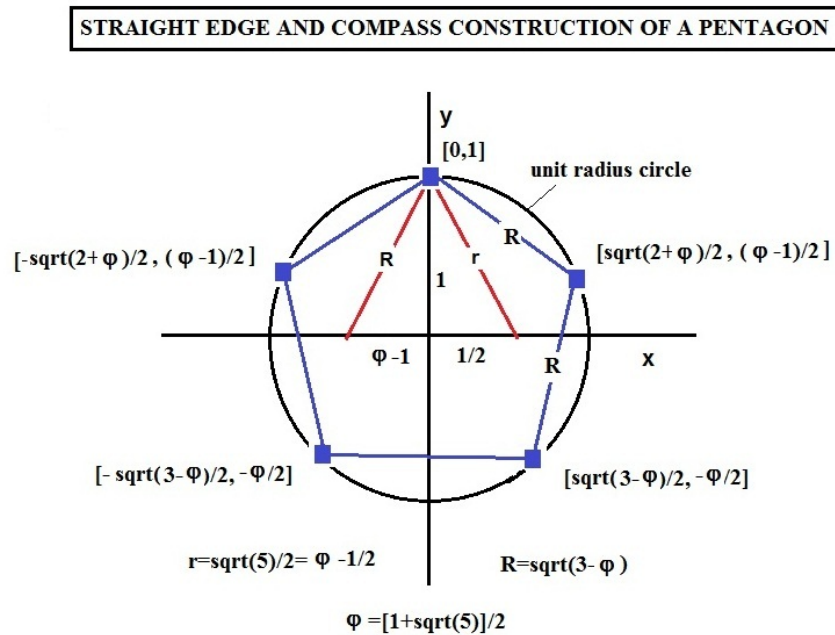
Here is the construction described in words-

(1)-Draw an x axis and place a circle $x^2+y^2=1$ upon it using a compass. The radius 1 can be any distance since unit lengths can be chosen at will.

(2)-Draw a vertical line x=0 through the circle center established by connecting the intersections of two larger circles centered on the intersection of the original circle with the x axis.

(3)-Bisect the x axis at x=1/2 to locate a new point [x,y[=[1/2,0]

(4)-Draw a new circle about [1/2,0] whose radius equals the distance from [1/2,0] to [0,1]. This radius has value $r=sqrt(5)/2=\phi-1/2$, where ϕ is the Golden Ratio.

(5)-The new circle will intersect the point [0,1] plus the negative x axis at [1-φ,0]=[-0.6180,0]. Reset the compass to a new radius R=sqrt(3-φ) which equals the distance from [0,1] to [1-φ,0].

(6)-Draw a new circle centered on [0,1] having radius R=sqrt(3-φ) .

(7)-This produces two intersections of the original unit radius circle in the first and second quadrant given by [±sqrt(2+φ)/2,(φ-1)/2]. These are the next two vertexes of the pentagon.

(8)-By centering the compass on these points and retaining the radius R, one can obtain the remaining two vertices. The remaining vertex points are located in the third and 4th quadrant at[±sqrt(3-φ)/2,-φ/2].

(9)-Finally, connecting these five vertexes with straight lines produces the desired regular polygon with edges shown in blue. Note that the distance R between neighboring vertices represents essentially the side lengths for a pentagon. All its vertices lie on the unit radius circle.

Since regular polygons have all their vertices lie an a given circle it suggests we can quickly determine the coordinates of these points in either polar or Cartesian coordinates The procedure for doing so starts with placing the first vertex at the polar coordinate location [r,ψ]=[r,c]. This vertex plus all the remaining ones are then located at-

$$[r,\psi]=[r, c+2\pi k/n] \quad \text{with} \quad k=0,1,2,\ldots,(n-1)$$

In Cartesian coordinates we have the vertices of the n sided polygon located at-

$$x=r \cos(c+2\pi k/n) \quad \text{and} \quad y=r \sin(c+2\pi k/n]$$

The relation between the circle radius r and the side-length s remains as s=2rcos(2π/n). One can easily program these x and y locations to produce pointplots and listplots of any n sided polygon. The following is such a graph for the heptadecagon of Gauss-

THE 17 SIDED HEPTADECAGON OF GAUSS SHOWING
VERTICES AND CONNECTING LINES

r=1

ψ=2π/17
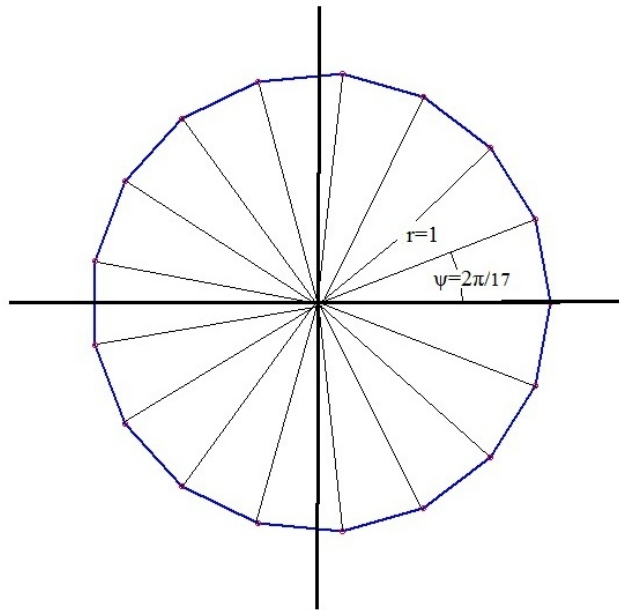
The vertices are shown as red circles and the connecting lines, representing the edges of the polygon, are in blue. We have chosen to place the first vertex at r=1 and ψ=0 and hence the constant in the angle c has been set to zero.

To magnify, rotate and translate any regular polygon is a straight forward procedure. We can write the transformed vertex coordinates as-
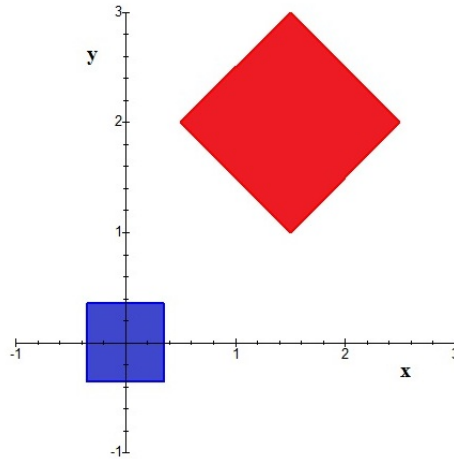
$$x=a +mr\cos(d+2\pi k/n) \quad and \quad y=b+mr\sin(d+2\pi k/n)$$

Here m is the magnification factor, d=(c+rotation angle) the rotation factor, and a and b the shift in x and y of the polygon center. By squaring and then adding the result, we get-

$$(x-a)^2+(y-b)^2=(mr)^2$$

This is just the shifted guiding circle of radius mr on which the vertices of the transformed polygon lie. As an example of such a transformation we have taken a square originally centered on the origin and translated it to x=3/2 and y=2, rotated it by 45º, and doubled its side-length. Here is the resultant graph-

The MAPLE math program we used to generate this graph is-

with(plots):
A:=listplot([seq([3/2+cos(-Pi/2+2*Pi*n/4),2+sin(-Pi/2+2*Pi*n/4)],n=0..4)],color=red,
numpoints=2000,view=[-1..3,-1..3],scaling=constrained,thickness=2):
B:=listplot([seq([0.5*cos(Pi/4+2*Pi*n/4),0.5*sin(Pi/4+2*Pi*n/4)],n=0..4)],color=blue,
numpoints=2000,view=[-1..3,-1..3],scaling=constrained,thickness=2):
display(A,B,title=`MAGNIFICATION,ROTATION,AND TRANSLATION OF A
SQUARE`);

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