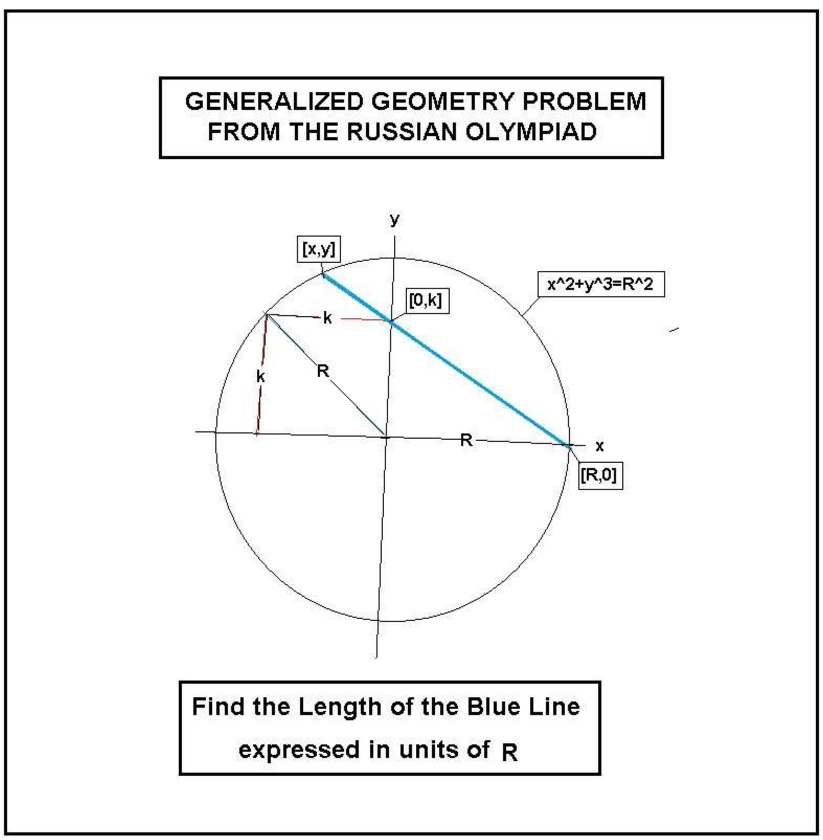
SIMPLE SOLUTIONS TO SOME MATH OLYMPIAD QUESTIONS

Many prestigious universities throughout the word require that those who are majors in math or science take a test known collectively as the MATH OLYMPIAD. The answers to many of the questions on this exam are found throughout the internet, but unfortunately usually are given in a manner not in the simplest form as expected by the examiners in order to judge the students math capability. We want in this article to answer some the Olympiad questions in the simplest possible manner relying as little as possible on earlier rote learning.

LENGTH OF A LINE IN A CIRCLE:

We begin with the Russian Math Olympiad problem depicted in the following picture-



One wants to find the length of the blue line inside a circle of radius R. The line contains the three points [x,y], [0,k], and [R,0]. Also ksqrt(2)=R. There are two triangles defined between the blue line and the x axis. From these we have that-

$$y=(R-x)/\sqrt{2}$$

Here x and y lie on the circle $x^2+y^2=R^2$. We can write, upon eliminating y, that-

$$3x^2-2Rx-R^2=0$$

This quadratic solves as $x=-R/3$ from which $y=4R/[3\sqrt{2}]$. The length of the blue line becomes-

$$L=\sqrt{(4R/3)^2+[4R/(3\sqrt{2})]^2}=4R/\sqrt{6}=1.632993..\, R$$

FINDING ALL ROOTS OF $x^k-1=0$ where k IS ANY POSITIVE INTEGER:

Here we see at once that-

$x=1$ for $k=1$

$x=+1$ or -1 for $k=2$

$x=+1$, $[-1+i\sqrt{3}]/2$, $[-1-i\sqrt{3}]/2]$ for $k=3$

One sees that the number of distinct roots equals k and that these lie on a unit circle separated by angles $2\pi/k$. The general solution for a given k is given by-

$$x=\exp(i2\pi n/k)=\cos(2\pi n/k)+I\sin(2\pi n/k) \quad \text{with } n=1,2,3,\ldots k$$

The Math Olympiad question found on the internet corresponds to k=5 and has the unique five solutions-

$$x=\cos(2\pi n/5)+i\sin(2\pi n/5)$$

Of these only x=1 has a real value while the remaining four are complex.

VALUE OF THE SUM OF DIFFERENT POWERS OF X:

Here we look at a problem found on the Oxford University Math Olympiad. It asks one to find the value of x which makes –

$$x^9-x^3=6$$

To solve it we first make the substitution $u=x^3$ to yield-

$$u^3=6-u \quad \text{or the equivalent} \quad (u^2-1)=6/u$$

Next we construct the following table-

u	6/u	u^2-1
1	6	0
2	3	3
3	2	8

We see column two and three match at u=2. Hence –

$$x=2^{(1/3)}= 1.259921021\ldots$$

EVALUATING THE PRODUCT OF AN IRRATIONAL NUMBER TAKEN TO A LARGE INTEGER POWER:

Here is a problem taken from the Harvard MATH Olympiad. It asks one to find-

$$N=[\mathrm{sqrt}(2)-1]^{12})$$

The simplest way to attack this problem is to first get rid of the minus sign in the expression. This produces –

$$N1=[\mathrm{sqrt}(2)-1]=1/[\mathrm{sqrt}(2)+1]$$

On squaring we get –

$$N2=1/[3+2\mathrm{sqrt}(2)]$$

This in turn produces-

N4=(N2)^2=1/[17+12sqrt(2) and N8=(N4)^2=577+408sqrt(2)

To get the desired answer we write-

N12=(N8*N4)=i/{[(577+408 sqrt(2)]*[17+12sqrt(2)]} On multiplying the denominator of this last expression we find the answer-

N12=1/[19601+12860sqrt(2)]=[19601-12860sqrt(2)]

PROBLEMS WHERE THE UNKNOWN INTEGER APPEARS IN THE EXPONENT:

Here a typical MATH OLIMPIAD problem looks as follows-

3^x=(7x+6)

, with x a positive integer.

To find its answer the typical well trained math major will note at once that this is the type of problem will involve the Lambert W function. This approach indeed will yield a correct solution but only after considerable effort. Alternatively one can simply look at the two sides of the equation and see when 3^x is greater than 7x+6 and when the reverse is true. A brief table will tell one where to find the correct value of x. For this problem such a table looks as follows-

x	3^x	7x+6
1	3	13
2	9	20
3	27	27
4	81	34

We see that the two sides match at x=3. This is the solution obtained with a minimum of effort.

The above examples have shown how one can quickly solve MATH OLYMPIAD problems using only the simplest mathematical manipulations. A good understanding of evaluating the powers of complex numbers, including knowledge of geometry and trigonometric functions, is all that is required.

U.H.Kurzweg
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Gainesville, Florida