PADE APPROXIMANTS

It is known that one may approximate the first N=M+L terms of an infinite series as the ratio of two polynomials. Such approximations are termed Padé Approximants. We have already shown in several earlier notes above that for the special cases of arctan(x) and ln(x) one can quickly obtain polynomial quotients by an elegant approach involving the integration of integrals containing Legendre polynomials. However, such an approach will not work for most other functions F(x), and one is thus forced to find the quotients by a brute force approach originally due to Henri Padé(1863-1953). We present here a quick development of the Padé Method.

Consider a function F(x) and its MacLaurin Series approximation S_N(x) involving the first N+1 terms-

$$F(x) \approx S_N(x) = \sum_{n=0}^{N} c_n x^n = c_0 + c_1 x + c_2 x^3 + ... + c_N x^N$$

Next assume that this truncated series can be represented by the quotient-

$$Q_{M,L}(x) = \frac{\sum_{n=0}^{M} a_n x^n}{1 + \sum_{n=1}^{L} b_n x^n}$$

where M and L are numbers to be specified. If one now collects the coefficients for the different powers of x, one obtains the hierarchy of algebraic equations-

$$\begin{aligned}
&a_0 = c_0 \\
&a_1 = c_0 b_1 + c_1 \\
&a_2 = c_0 b_2 + c_1 b_1 + c_2 \\
&a_3 = c_0 b_3 + c_1 b_2 + c_2 b_1 + c_3 \\
&etc
\end{aligned}$$

One stops with a_M and b_L and requires that the MacLaurin series expansion of $Q_{M,L}$(x) agree with the series for F(x) up to the M+L power of x and the terms a_{M+1} and b_{L+1} (and higher) are set to zero. This provides sufficient information to solve for all a_ns and b_ns to obtain the unique Padé Approximate Q $_{M,L}$(x) .

Let us demonstrate for the exponential function

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$

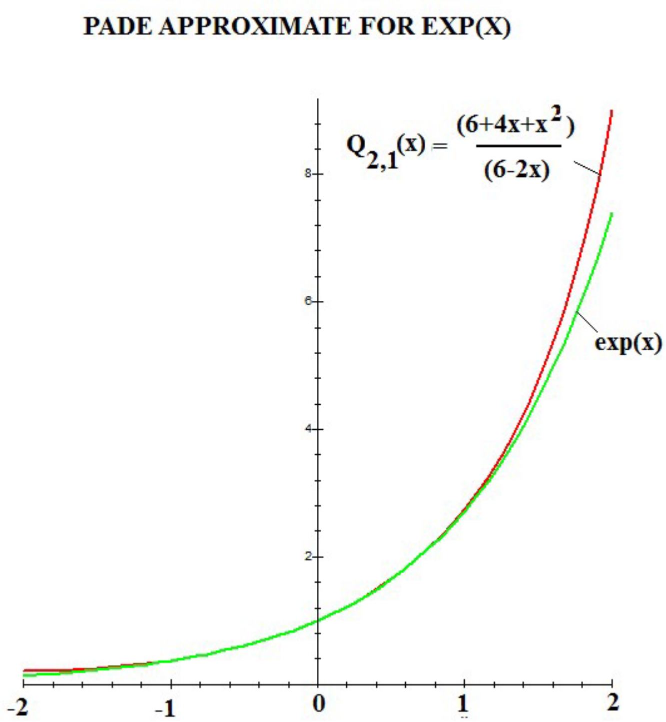
Using a Q $_{2,1}$(x) approximation. One finds-

$$a_0 = 1$$
$$a_1 = b_1 + 1$$
$$a_2 = 0 + b_1 + (1/2!)$$
$$0 = 0 + 0 + (1/2!)b_1 + (1/3!)$$

This produces the Padé Approximant –

$$\exp(x) \approx Q_{2,1}(x) = \frac{6+4x+x^2}{6-2x} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{18}x^4 + \ldots$$

which is seen to be accurate through the x^3 term. A graph of Q $_{2,1}$(x) versus exp(x) in the range -2<x<2 looks like this-



The approximation between x= -1 and +1 is seen to be excellent with expected departures occurring beyond this range. Improvements can be gotten by increasing the values of M and L. In general a function F(x) can have multiple Pade Approximates. One can use routines from canned math programs to quickly

determine the quotients $Q_{M,L}(x)$. For example our MAPLE program shows that a Pade Approximate $Q_{3,4}(x)$ for sin(x) is gotten as follows and yields-

with(numapprox): T:=pade(sin(x), x, [3,4]); series(T,x, 9);

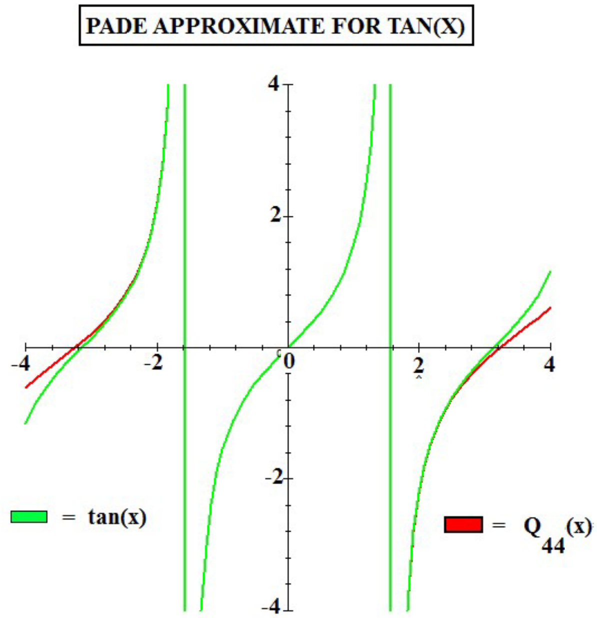
$$\sin(x) \approx Q_{3,4}(x) = \frac{x + \frac{31}{294}x^3}{1 + \frac{3}{49}x^2 + \frac{11}{5880}x^4} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^9)$$

good through x^7. Note that the odd symmetry of sin(x) is maintained in the powers of x appearing in the polynomials making up this last quotient.

One of the more important applications of Pade Approximates is their use in determining the location of singularities of functions when only the first few terms in a functions expansion are known. This is especially true for certain theoretical physics calculations. Let us demonstrate by looking at the function F(x)=tan(x) which has singularities at x=±π/2, ±3π/2, etc. The Pade Approximate $Q_{4,4}(x)$ reads-

$$\tan(x) \approx Q_{4,4}(x) = \frac{x - (2/21)x^3}{1 - (3/7)x^2 + (1/105)x^4}$$

The vanishing of the denominator in this quotient indicates that the function has singularities at x=±1.5712.. . Recalling that π/2=1.5707…, one sees that the quotient very nicely predicts the location of the singularities of tan(x) nearest to x=0. Comparing the following graph of $Q_{44}(x)$ with tan(x) shows excellent agreement over the range -3<x<3.



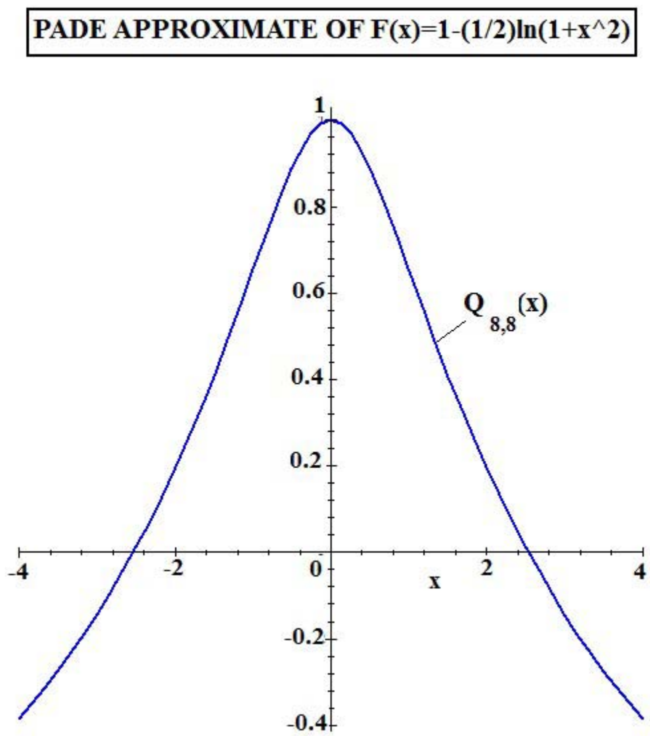
Finally consider the function –

$$F(x)=1-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^6}{6}-\frac{x^8}{8}+O(x^{10})=1-\frac{1}{2}\ln(1+x^2)$$

For this case, our MAPLE Pade program yields-

$$Q_{8,8}(x)=\frac{1+\frac{3}{2}x^2+\frac{15}{28}x^4-\frac{1}{42}x^6-\frac{13}{840}x^8}{1+2x^2+\frac{9}{7}x^4+\frac{2}{7}x^6+\frac{1}{70}x^8}$$

Note that the even symmetry of F(x) in this case which means only even powers of x appear in the quotient. In the range |x|<4 this quotient (whose graph follows) gives an excellent approximation to F(x)-



At the same time it should be pointed out that such more complicated quotients involving larger values of M and L can become sufficiently cumbersome so that often it is simpler to just study the behavior the original function as x is varied.

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