CONTRUCTING PADE APPROXIMATES

A standard Maclaurin Series can be written as-

$$f(x)=\sum_{k=0}^{N}c_k x^k = c_0 + c_1 x + c_2 x^2 + ... + c_N x^N \quad with \quad c_k = f(0)^{[k]}/k!$$

As first pointed out by the Frech mathematician Henri Pade(1863-2053) this expansion can also be approximated by the polynomial quotient-

$$R(x,m,n)=\frac{P(x,m)}{Q(x,n)}=\frac{\sum_{k=0}^{m}c_k x^k}{1+\sum_{k=1}^{n}c_k x^k}=\frac{a_o + a_1 x + a_2 x^2 + ...a_m x^m}{1+b_1 x + b_2 x^2 + b_3 x^3 + ...b_n x^n}$$

Most advanced math computer programs call R(x,m,n) the **Pade Approximant** with a shorthand designation of pade(f(x),x,[m,n]). The evaluation of a_k and b_k coefficients are obtained by evaluating the identity-

f(x) Q(x,n)=P(x.m)

by setting the coefficients of x^k to zero. The larger m and n are taken the more accurate an approximation for f(x) will become. To find unique values for a_k and b_k will require m+n+1 algebraic equations and so f(x) must be expanded out to k=n+m starting with k=0.

We want in this note to develop several different Pade Approximates. Starting with one of the simplest forms, consider f(x)=exp(x) with m=n=1. This produces-

$$(1+x+\frac{x^2}{2})(1+b_1 x)=(a_0 + a_1 x)$$

and its three equations-

$a_0=1$, $a_1=b_1+1$, and $(1/2)+b_1=0$

The Pade Approximate becomes-

$$\text{pade(exp(x),x,[1,1])}=\frac{1+\frac{x}{2}}{1-\frac{x}{2}}$$

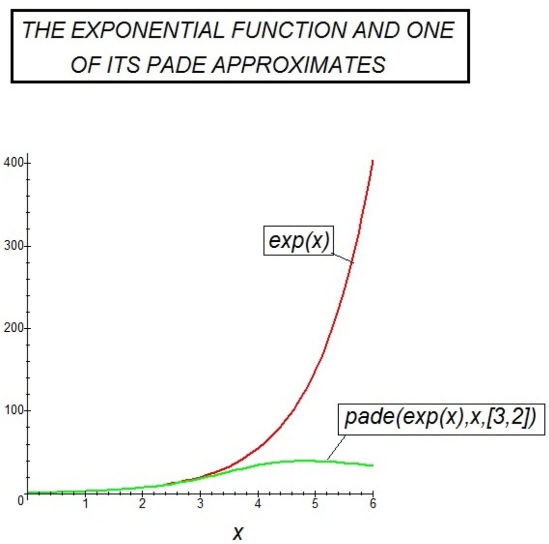
Note that m and n determine the highest powers of x present in P(x,m) and Q(x,n), respectively. Also the Maclaurin series contains x=m +n as its highest power. We next improve this pade approximation by considering m=3 and n=2. This produces-

$$(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120})(1+b_1x+b_2x^2)=(a_o+a_1x+a_2x^2+a_3x^3)$$

On solving for the coefficients we find-

$$pade(\exp(x),x,[3,2])=\frac{1+\frac{3}{5}x+\frac{3}{20}x^2+\frac{1}{60}x^3}{1-\frac{2}{5}x+\frac{1}{20}x^2}$$

A plot of exp(x), and pade(exp(x),x,[3,2]) follows-

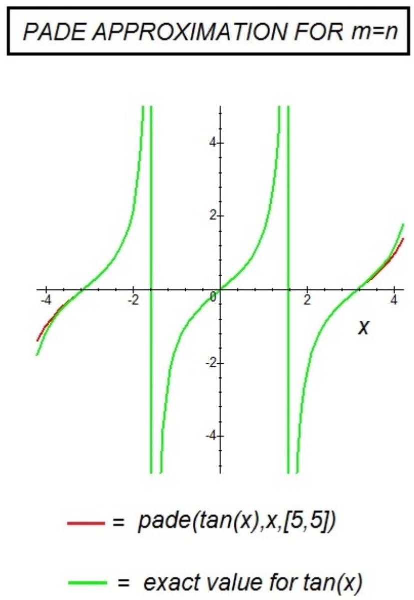


We see that the pade approximate is in good agreement with exp(1) up to about x=3. To get agreement above this value of x will require larger values for n and m.

Here is a pade approximate for the tangent function showing its singularities at (2n-1)π/2-

pade(tan(x),x,[5,5]) = $\frac{x}{15}\left\{\frac{945-105x^2+x^4}{63-28x^2+x^4}\right\}$

The graph follows-



A good estimate for the first infinity of tan(x) is found by solving $63-12x^2+x^4=0$. Its smallest root equals 1.570807884…. This is very close to $\pi/2=1.570796327$…. Note the approximation is very close to tan(x) over the range $-3.5<x<3.5$. A most interesting result also found is that our earlier use of the TLK Method for approximating the values of all trigonometric functions(see the April 11, 2020 article-https://mae.ufl.edu//KTL-UPDATE.pdf) yields a function T(5) identical with the above pade(tan(x),x,[5,5]) result. This is a rather amazing result considering that a completely different approach involving Legendre Polynomials was used to find the given quotient approximation T(5).

We continue on to find an approximation for π using the function 4pade(arctan(π/4),x,[50,50]). It produces the 38 digit accurate result-

$$\pi=3.14159265358979323846264338327950288 41\ldots$$

in a split second.

Consider next an approximation to ln(2) using pade(ln(1+x),x,[12,12];. It produces the 17 digit accurate result-

$$\ln(2)=0.69314718055994530\ldots$$

when x=1.

Note that m and n need not always be equal in pade approximations. I usually prefer to look at cases where m≥n. Also the larger n and m become the closer one will get to an exact answer. Symmetry of the function f(x) also plays a role in any pade approximate. The quotient must have the same symmetry as the function f(x). One sees this clearly in the above case of pade(tan(x),x,[m,n].

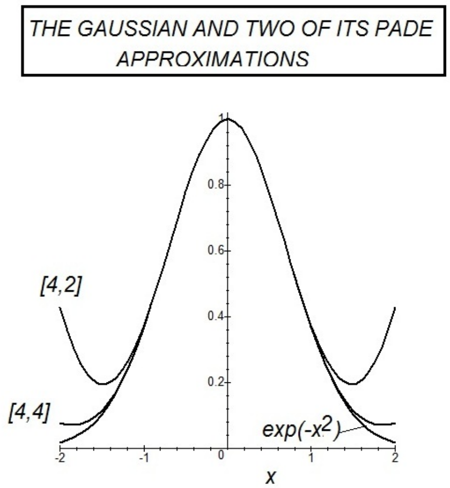
As a last example let us look at the Gaussian f(x)=exp(-x^2) for several different cases of[m,n]. It being an even function means that all powers of x in the pade approximate will be even. Here are three of such quotients-

$$\text{pade(exp(-x^2),x,[2,2])}=\frac{1-\frac{1}{2}x^2}{1+\frac{1}{2}x^2}$$

$$\text{pade(exp(-x^2),x,[4,2])}=\frac{1-\frac{1}{3}x^2+\frac{1}{6}x^4}{1+\frac{1}{3}x^2}$$

$$\text{pade(exp(-x^2,x,[4,4])}=\frac{1-\frac{1}{2}x^2+\frac{1}{12}x^4}{1+\frac{1}{2}x^2+\frac{1}{12}x^4}$$

A plot of two of these and the Gaussian follow-



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