FINDING THE EQUATION OF A PLANE CONTAINING THREE POINTS

Have you ever wondered why a table or chair placed on the ground sometimes needs a wedge under one of its legs while a camera tripod never does? The answer is that it takes just three points(three legs) to uniquely define a plane with the ground be it level or not and unless the fourth point given by the end of the remaining leg is just the right height the table or chair will wiggle. I have encountered this problem numerous times at outdoor cafes throughout the world and am always prepared to stuff something under the fourth leg. Lets look at this problem from a mathematical viewpoint. Start with the equation of a general plane in space. It has the unique form-

$$Ax + By + Cz = 1$$

whereA, B, and C are constants to be determined. To find the values of these constants one needs to only specify three points in space. Call these points-

$$P(x_1, y_1, z_1), \quad P(x_2, y_2, z_2), \; and \; P(x_3, y_3.z_3)$$

Substituting these values into the equation for the plane, one obtains three linear algebraic equations which can be written in matrix form as-

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In abbreviated form this matrix equation reads MF=G. Provided the coefficient matrix M is not singular, one can invert M to get the solution-

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = M^{-1}G$$

and the plane is thus found. Note that a fourth point is not necessary to define this plane. If the fourth point is to lie on the same plane then the extra condition-

$$Ax_4 + By_4 + Cz_4 = 1$$

will be required. This is recognized to be a Diophantine equation with multiple solutions possible.

Let us next carry out a specific solution. Take the points $P_1(1,2,-1)$, $P_2(0,1,3)$, and $P_3(2,1,-1)$. Here we find using MAPLE that-

$$M = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & 1 & -1 \end{bmatrix} \quad and \quad M^{-1} = \begin{bmatrix} -2/5 & 1/10 & 7/10 \\ 3/5 & 1/10 & -3/10 \\ -1/5 & 3/10 & 1/10 \end{bmatrix}$$

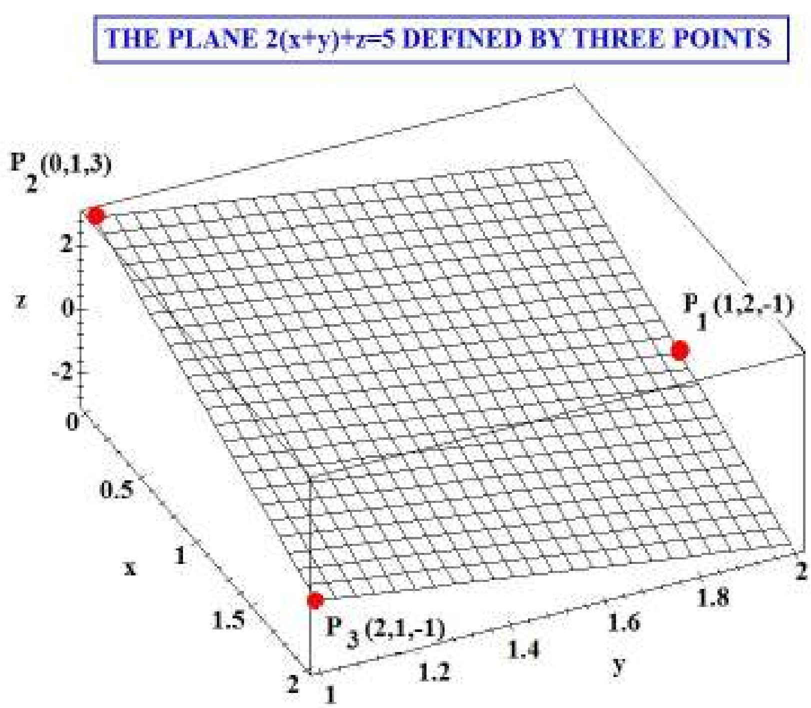
with det(M)=10. Thus we have the unique solution-

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}$$

yielding the plane-

$$2x + 2y + z = 5$$

A 3d plot of this plane, and the three points used to define it, follow-



To place a fourth point on this plane, we could choose $P_4(0,2,1)$ or $P_4(2,2,-3)$ among an infinite number of other possibilities. To get the fourth leg of a chair or table just right is not that easy. In my woodworking shop I usually accomplish the coplanar positioning to sub-millimeter accuracy using a long level.