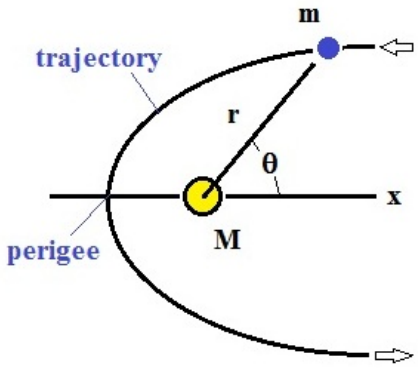
CALCULATING PLANETARY ORBITS ABOUT THE SUN

One of the earliest and most significant contributions of Newtonean mechanics was the verification of Kepler's three laws of planetary motion. We want here to briefly go through the mathematics which allowed Newton to derive the properties of planetary motion about the sun. Our starting point is the following schematic-

PLANETARY TRAJECTORY ABOUT THE SUN



We have here a planet of mass m moving in an orbit about the sun of much larger mass M. The polar coordinates used are the radial distance r between the centers of the two masses and θ the angle r makes with respect to the symmetry axis x. In terms of Newton's second law and the universal law of gravitation one has the two governing equations-

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} \quad and \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

The second of these equations is just a conservation of angular momentum statement and is equivalent to saying –

$$h = r^2\dot{\theta} = const.$$

This last result also says that the area swept out in unit time by a planet moving about the sun is h/2 (Kepler's 2nd Law). Eliminating v_θ from the first equation above we have-

$$\ddot{r} = \frac{h^2}{r^3} - \frac{GM}{r^2}$$

Integrating once we get-

$$\frac{1}{2}(\dot{r})^2 + \frac{h^2}{2r^2} - \frac{GM}{r} = Const.$$

If we multiply this result by m we have the conservation of energy statement -

$$\frac{m}{2}(v_r{}^2 + v_\theta{}^2) - \frac{GMm}{r} = mE$$

where E is the constant total energy of the planet per unit mass and –GMm/r the potential energy. Dividing this equation by m/2 and re-substituting for v_θ yields-

$$v_r{}^2 = \frac{2GM}{r} - \frac{h^2}{r^2} + 2E$$

Next letting u=1/r and noting that v_r=-h(du/dθ), one finds-

$$(\frac{du}{d\theta})^2 = \beta^2 - (u-\alpha)^2$$

, where $\alpha=GM/h^2$ and $\beta=$ sqrt[$\alpha^2+(2E/h^2)$]. We can integrate this last result once more to get-

$$\theta = \int \frac{du}{\sqrt{\beta^2 - (u-\alpha)^2}}$$

Now recalling from the integral tables that-

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}(\frac{x}{a})$$

we find that the planet trajectory is given by the conic section-

$$r = \frac{\Delta}{1 - e\cos(\theta)}$$

where $\Delta=1/\alpha^2=h^2/GM$ and $e=\beta/\alpha=$sqrt[$1-2h^2|E|/(GM)^2$] the eccentricity of the conic section defined by this last equation. The constant in the angle has been adjusted so as to make the x axis a symmetry axis and the near point(perigee) from the central mass M occur when θ=π.

When e<1, this trajectory will be an ellipse (Kepler's 1st Law) whose eccentricity is given by-

$$e = \frac{\sqrt{A^2 - B^2}}{A} = \sqrt{1 - \frac{2 \mid E \mid \Delta^2}{h^2}} < 1$$

,where A and B are the semi-major and semi-minor axes of the ellipse. A little manipulation shows that $A=(r_a+r_p)/2=\Delta/(1-e^2)$, where r_a and r_p are the largest and smallest distance from the sun the planet finds itself at during its trajectory .

To calculate the period τ of the elliptic orbit, we recall that the area of an ellipse equals $\pi AB = \pi A^2 \text{sqrt}(1-e^2)$ and that the area swept out per time is h/2 (from Kepler's 2nd law). Thus-

$$\tau = \frac{2\pi A^2 \sqrt{1 - e^2}}{h}$$

But from the energy equation given earlier we know at perigee $v_r=0$ so that-

$$v_\theta{}^2 = (\frac{h}{r_p})^2 = \frac{2Gm}{r_p} + 2E$$

Therefore we have that the angular momentum term must equal-

$$h = \sqrt{2GMr_p + 2Er_p{}^2} \quad with \quad r_p = \frac{h^2}{(GM)(1+e)} \quad and \quad A = \frac{h^2}{GM(1-e^2)}$$

Combining these results then produces-

$$h^2 = AGM(1 - e^2)$$

Substituting this value for h into the τ equation yields the desired result-

$$\tau^2 = \frac{4\pi^2 A^4 (1 - e^2)}{AGM(1 - e^2)} = \frac{4\pi^2}{GM} A^3$$

This is Kepler's famous third law of planetary motion which says the square of a planet's orbit period τ is proportional to the third power of the semi-major axis A of its elliptical path. For the special case of a circular orbit of radius A=R one has an orbit period equal to-

$$\tau = 2\pi \sqrt{\frac{R^3}{GM}}$$

If we look at the earlier given energy equation as a potential well problem, we can see that if a mass m has total energy E=0 at $r=\infty$, then the kinetic energy at perigee will be-

$$\frac{mh^2}{2r_p^{\,2}} = \frac{GMm}{r_p}$$

This is equivalent to saying that the speed at perigee will be-

$$v_\theta = \sqrt{\frac{2GM}{r_p}}$$

This last result is recognized as the escape velocity from mass M by a smaller mass m. For the case of escaping from the earth to infinity one has $GM=gR^2$ and $r_p=R$ the earth radius. Hence to escape from the earth's surface will require an effective speed of-

$$v_\theta = \sqrt{2gR} = \sqrt{2 \cdot 9.8066 \cdot 6.371} = 11.2 km/\sec = 36.7 \text{ thousand ft/sec}$$

To get out of the solar system will require a much higher speed than this because of the sun much larger mass. Note that the speed of a satellite in a near earth circular orbit will be less by a sqrt(2). In other words 7.9 km/sec or about 26 thousand foot per second for the earth.

When e>1 the trajectory will be in form of a hyperbola. Casting the $r=\Delta/[1-e\cos(\theta)]$ equation into Cartesian coordinates when e>1 produces a standard hyperbola-

$$\frac{(x+|b|)^2}{e|b|^2} - \frac{y^2}{a|b|e^2} = 1 \quad where \quad |b| = \frac{a}{(e^2-1)} > 0$$

and 'a' is the distance from the directrix , which crosses the negative part of the x axis at right angles, to the center of M. Here $x=a+r\cos(\theta)$, $y=r\sin(\theta)$, and the eccentricity is $e=r/[a+r\cos(\theta)]$.