MTH ORDER POLYNIMIALS AND CORRESPONDING SEQUENCES

If one considers the second order polynomial-

$$P(n)=(n+1)^2$$

at integer values 1,2,3,4,.., one can generate the following table-

n	P(n)
0	1
1	4
2	9
3	16
4	25

That is, P(n) represents the sequence-

S(n)={1,4,9,16,25,..n^2)

, whose elements represent the square of the integers. In a similar manor, other polynomials P(n) will have associated with them an infinite sequence S(n) whose elements equal P(n) evaluated at integers. Thus the polynomial –

$$P(n)=(n/2)+(n^2/2)$$

yields the sequence-

S(n)=1,3,6,10,15,21,28,..)

The elements here represent the sum of the integers up through n. We want in this note to determine the polynomials associated with various integer element sequences and visa versa.

Let us start with the sequence –

S(n)={1,2,4,7,11,…}

and carry out the following difference operation-

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1   2   4   7   11
  1   2   3   4
    1   1   1
```

We see that the second difference is constant meaning that the corresponding polynomial must have the quadratic form -

$$P(n)=a_0+a_1 n +a_2 n^2$$

Solving for the constants alpha, we get the polynomial–

$$P(n)=1+(n/2)+(n^2/2)$$

with-

$$S(n)=\{1,2,4,7,11,..\}$$

Here the n=100 element in the sequence will be 5051.

Consider next the sequence-

$$S(n)=\{1,5,14,30,55,91,..\}.$$

Here the differences yield-

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1   5   14   30   55   91
  4   9    16   25   36
    5   7     9   11
      2     2    2
```

The third difference is a constant so the corresponding polynomial will be of the form-

$$P(n)=a_0+a_1n+a_2n^2+a_3n^3$$

To solve for the a_ns we need to evaluate the simultaneous equations-

$$1=a_0$$
$$5=a_0+a_1+a_2+a_3$$
$$14=a_0+2a_1+4a_2+8a_3$$
$$30=a_0+3a_1+9a_2+27a_3$$

These yield $a_0=1$, $a_1=13/6$, $a_2=3/2$, and $a_3=1/3$.
Thus we have –

$$P(n)=(1/6)[6+13n+9n^2+2n^3]$$

with-

$$S(n)=\{1,5,14,30,55,91,140,204,..\}$$

Note that S(100)=348551. In looking at neighboring terms we see 5-1=2^2,14-5=3^2, 30-14=4^2 etc..
This implies that-

$$S(n+1)=S(n)+(n+2)^2$$

So that we are dealing with the sum of the squares of the first n integers.

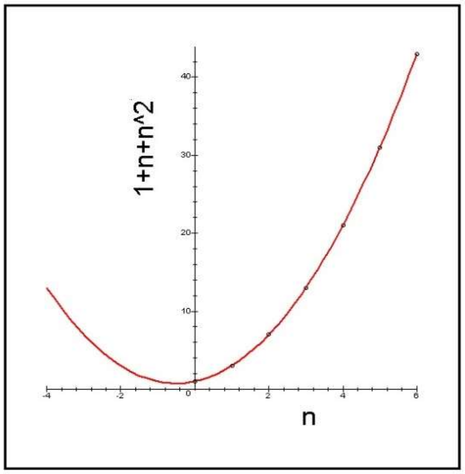
Take next the sequence-

$$S(n)=\{1,3,7,13,21,31,..\}$$

Taking differences between neighbors, shows that the corresonding polynomial is quadratic. Evaluating its coefficients produces-

$$P(n)=1+n+n^2$$

A plot of P(n) versus the elements S(n) follows-



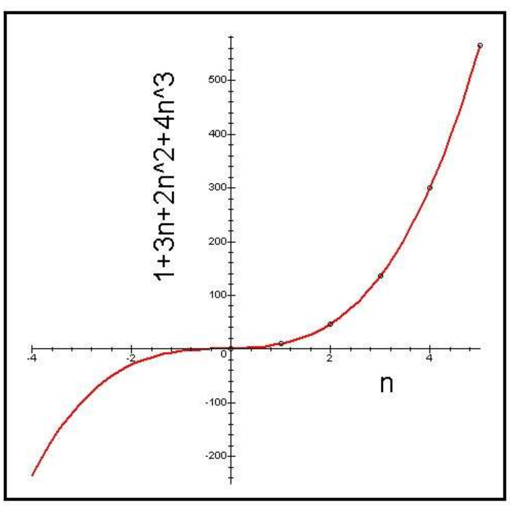
All of the above examples have started with S(n) to produce the corresponding polynomial P(n). The reverse can also be carried out and is often much easier. Consider the third order polynomial-

$$P(n)=1+3n+2n^2+4n^3$$

Here P(0)=1, P(1)=10,P(2)=47,P(3)=136,P(4)=301. So the corresponding infinite sequence becomes-

$$S(n)=\{1,10,47,136,301,..\}$$

A plot of P(n) in n=-4..5 and the corresponding points S(n) for n in 0<n<5 follow-



Note that if one allowed negative integers in our sequence then S(n) would contain the extra elements P(-1)=-4,P(-2)=-29,P(-3)=-98, etc..

Finally consider the quadratic polynomial $P(n)=(1/2)[n+n^2]$.Here we have-

$P(1)=1$

$P(2)=1+2=3$

$P(3)=1+2+3=6$

$P(4)=1+2+3+4=10$

$P(5)=1+2+3+4+5=15$

So that $S(n)=\{1,3,6,10,15,..\}$. The elements in S(n) thus just represent the sum of subsequent integers added up through n. That is $P(100)=5050$.

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