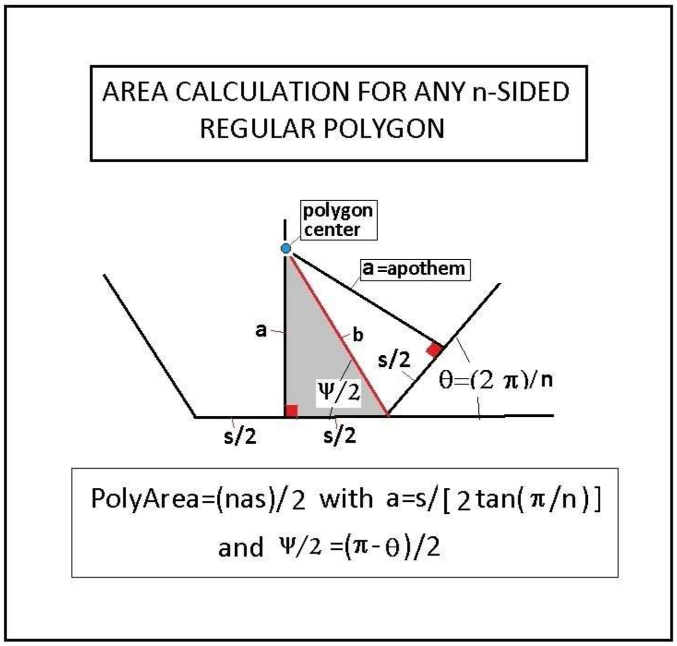
AREAS OF REGULAR n-SIDED POLYGONS

We wish in this note to show how one determines the areas of regular polygons involving the use the product of apothem and circumference. Also we look in detail at the area solution for a pentagon expressing things as a function of the Golden Ratio and root five.

We start with the following sketch-



The figure shows three sides of a regular n sided polygon plus two apothems of length 'a' each meeting at the polygon center. The exterior angle equals-

$$\theta = 2\pi/n$$

and the larger angle of the shaded right triangle is-

$$\psi/2=(\pi/2)[1-(2/n)]$$

Taking the tangent, we find the apothem-

$$a=(s/2)\tan(\psi/2)=s/[2\tan(\pi/n)]$$

The total area A of the regular polygon will thus be 2n times that of the shaded triangle. Thus we have-

$$A=ans/2=[\text{apothem x polygon perimeter}]/2$$

Substituting the expanded form of ‘a’ into this last expression, yields-

$$A=(ns^2)/[4 \tan(\pi/n)]$$

Dividing by s^2 produces the non-dimensional equivalent form-

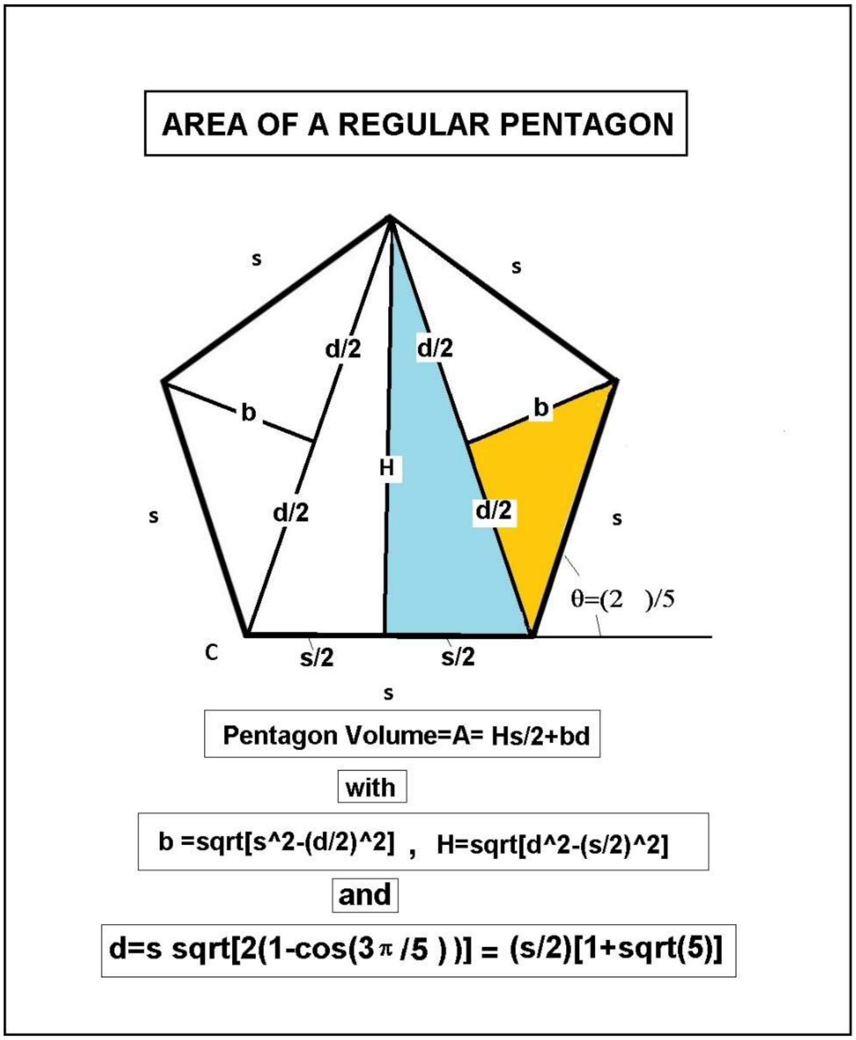
$$[A/s^2]=n/[4 \tan(\pi/n)]$$

This result is valid for any regular polygon ranging from n=3 to infinity. Note that for very large n the polygon approaches a circle with diameter D=2a=ns/π. That is, the polygon becomes a circle with its circumference equal to sn and radius r= ‘a’. Here is a table of the values of A/s^2 for some of the better known regular polygons-

n	Polygon Name	n/[4 tan(π/n)]
3	Equilateral Triangle	0.433012
4	Square	1.000000
5	Pentagon	1.720477
6	Hexagon	2.598076
8	Octagon	4.828427
10	Decagon	7.694208
12	Dodecagon	11.196152
20	Icosagon	31.568757

Note that the polygon area goes up by a factor of four when n is doubled and n is kept large.

Although the above tan formula offers the easiest way to calculate a regular polygon area, there are other approaches which offer the results as roots of integers. This approach is particularly effective for regular pentagons whose areas involve the Golden Ratio and the root of five. Let us develop this approach. This time we start with the following modified diagram-



We notice that the interior angle about each of the five pentagon vertexes equals $(\pi-2\pi/5)=3\pi/5$ rad=108 deg. So applying the law of Cosines to the large interior oblique triangle yields-

$$d=s\ \mathrm{sqrt}[\ 2(1-\cos(3\pi/5))]\ =s\ \mathrm{sqrt}(2.618033989)=s\varphi$$

, where φ =[1+sqrt(5)]/2 =1.618033989..is the Golden Ratio. Next , by the Pythagorean Formula, we find[

$$b=\mathrm{sqrt}[s^2-(d/2)^2]\ =(s/2)\ \mathrm{sqrt}(4-\varphi^2)$$

and –

$$H=\mathrm{sqrt}[d^2-(s/2)^2]=(s/2)\ \mathrm{sqrt}(4\varphi^2-1)$$

Using the above values, we find the pentagon area A to be-

A=bd+sH/2=(s^2/4)(2φ+1)sqrt(4φ^2-1)

That is-

A/s^2=(1/4)[(2φ) sqrt(4-φ^2)]+sqrt(4φ^2-1)]=1.7204774

This result is identical with the value of A/s^2 obtained earlier using the apothem–perimeter approach. Finally we can also eliminate φ by setting it to [1+sqrt(5)]/2. After some manipulations it produces-

A/s^2= (1/4) sqrt[5(5+2 sqrt(5))]

We can also express the regular area A of any other regular polygon in terms of roots of integers by converting the tangent term n/[4 tan(π/n)] to roots of integers. Without showing the additional steps, here are integer root representations for n=3, 4, 5, 6, 8, 10,12, and 20-

n-number of Polygon Sides	A/s^2 as Integer Roots	A/s^2 as Integer Fraction
3	sqrt(3)/4	0.433012
4	1	1.000000
5	sqrt[5(5+2sqrt(5))]/4	1.720477
6	(3/2)sqrt(3)	2.598076
8	2[1+sqrt(2)]	4.828427
10	5sqrt[5+2sqrt(5)]/2	7.694209
12	3[2+sqrt(3)]	11.196152
20	5{1+sqrt(5)+sqrt[5+2sqrt(5)]}	31.568757

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