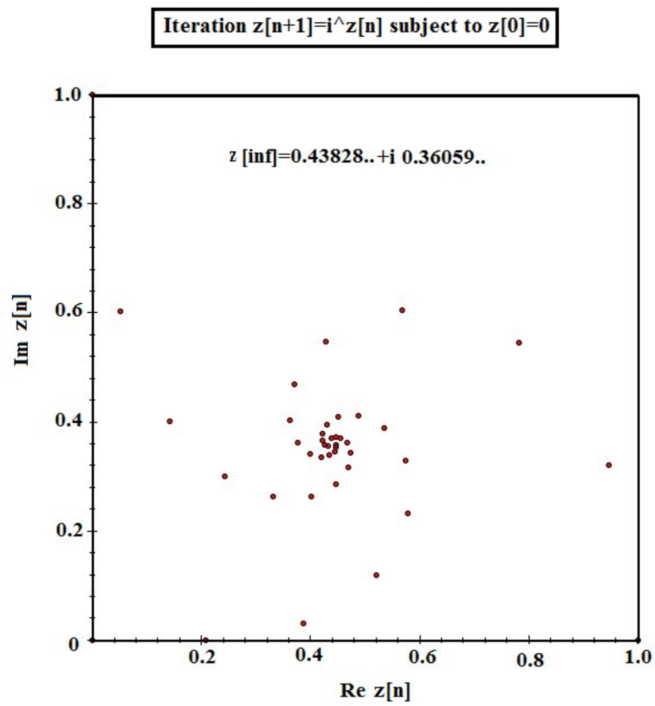
POWER TOWERS AND THE TETRATION OF NUMBERS

Several years ago while constructing our newly found hexagonal integer spiral graph for prime numbers we came across the sequence-

$$S = \{i, i^i, i^{i^i}, ...\},$$

Plotting the points of this converging sequence out to the infinite term produces the interesting three prong spiral converging toward a single point in the complex plane as shown in the following graph-



An inspection of the terms indicate that they are generated by the iteration-

$$z[n+1] = i^{z[n]} \quad subject \quad to \quad z[0] = 0$$

As n gets very large we have z[∞]=Z=α+iβ, where α=exp(-πβ/2)cos(πα/2) and β=exp(-πβ/2)sin(πα/2). Solving we find –

Z=z[∞]=0.4382829366 + i 0.3605924718

It is the purpose of the present article to generalize the above result to any complex number z=a+ib by looking at the general iterative form-

$$z[n+1]=(a+ib)^{z[n]} \text{ subject to } z[0]=1$$

Here N=a+ib with a and b being real numbers which are not necessarily integers. Such an iteration represents essentially a tetration of the number N. That is, its value up through the nth iteration, produces the power tower-

$${}^{n}Z = Z^{Z^{Z^{Z}}} \quad \text{with n-1 zs in the exponents}$$

Thus-

$${}^{4}2 = 2^{2^{2^{2}}} = 2^{16} = 65536$$

Note that the evaluation of the powers is from the top down and so is not equivalent to the bottom up operation 4^4=256. Also it is clear that the sequence $\{{}^{1}2, {}^{2}2, {}^{3}2, {}^{4}2, ...\}$ diverges very rapidly unlike the earlier case $\{{}^{1}i, {}^{2}i, {}^{3}i, {}^{4}i, ...\}$ which clearly converges.

The simplest way to check whether the z[n+1] iteration converges or diverges is to run the iteration for a given complex number N=a+ib to a large value n and then look at the quotient-

$$Q[n] = \frac{1}{\lim n >> 1} abs\left\{\frac{z[n+1]}{z[n]}\right\}$$

If the quotient equals one then we have convergence, otherwise not.

Consider now the case N= sqrt(2) with z[0]=1. Here z[1]=sqrt(2), z[2]=$= \sqrt{2}^{\sqrt{2}}$, $z[3] = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ $etc.$. In this case we find Q[50]=1.000000001 with z[50]=1.999999993. So the sqrt(2) tower clearly converges to the finite value of z[∞]=2. Note however that sqrt(3) and other higher numbers all diverge.

Consider next the general complex number N=a +ib with z[0]=1. Iterations for this generic case read-

$$\text{z[1]=a+ib, } z[2] = (a+ib)^{(a+ib)}, \text{ } and \text{ } z[3] = (a+ib)^{(a+ib)^{(a+ib)}}, etc.$$

For a given a and b the z[n] sequence will either converge when the Q[n] criterion is met or diverge if not met. Let us assume for the moment that the convergence condion is met. Then we have that –

$$Z = z[\infty] = (a+ib)^{Z}$$

This equation may be rewritten as –

$$Z\exp\{-\ln(a+ib)Z[\infty]\} = 1$$

which, on multiplying by –ln(a+ib), leads to-

-ln(a+ib)Zexp(-ln(a+ib)Z=-ln(a+ib)

It is known that the Lambert Function W(x) is defined as W(x)expW(x)=x. Comparing we obtain the value-

$$Z = z[\infty] = \frac{W(\ln(\frac{1}{a+ib}))}{\ln(\frac{1}{a+ib})}$$

This value will only be good for conditions where the Q[n] criterion is met. So for N=sqrt(2) we find Z=W{ln(1/sqrt(2))}/ln(1/sqrt(2))=2. However for N=2 the iteration clealy blows up while the Lambert Function result yields the wrong answer of Z=1.771.

Consider three more power towers. One of these is-

$$(\frac{1}{e})^{(\frac{1}{e})^{(\frac{1}{e})}}$$

Its iterations go as z[1]=0.36787, z[2]=0.69220, z[3]=0.50047, with z[40]=0.567432905. The Q[n] criterion is satisfied so that we may use the the Lambert Function to predict the convergence point. It turns out to be precisely Z=W(1)=0.567143290409783873000 .

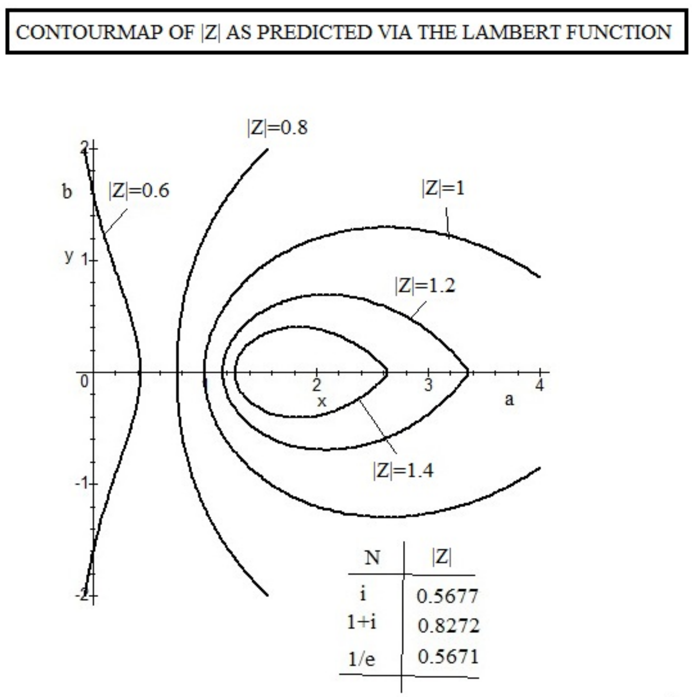
The power tower-

$$(\frac{1}{2})^{(\frac{1}{2})^{(\frac{1}{2})}} \qquad \text{converges to } \frac{W(\ln(2))}{\ln(2)} = 0.6411857444$$

Also the power tower-

$(1+i)^{(1+i)^{(1+i)^{(1+i)}}}$ converges to Z=$\dfrac{W(\ln(1/(1+i)))}{\ln(1/(1+i))}=0.6410264786+i0.5236284611$

Finally one can get a good idea for the value of Z=x[∞] for those N=a+ib where the iteration indicates convergence. This is achieved by looking at a contour map of the Lambert Function solution applicable typically when the magnitude of N=sqrt(a^2+b^2) does not become too large. Here is the contour map-



N	|Z|
i	0.5677
1+i	0.8272
1/e	0.5671

The value of |Z| for most of the N=a+ib numbers discussed above agree with the graph including the fact that for N=sqrt(2) we hit a value of |Z|=2. For values of N with larger a and b the present graph fails as it does, for instance, when N=3, 4, 5, etc. In that case the Q[n] criterion fails and the power tower has infinite value.

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