

PRIME, COMPOSITE, AND PERFECT NUMBERS

Several month ago while examining the properties of integers, we came up with a new way to classify various integers N via a quotient defined as the ratio of the sum of all divisors of a number , excluding N and one, divided by the number. That is-

$$f(N)=[\sigma(N)-(N+1)]/N$$

where $\sigma(N)$ is the sigma function of the number N . We term this quotient the **Number Fraction**. As found in books on number theory, one can express the sigma function occurring in this expression as-

$$\sigma(N) = \prod_{k=1}^n \frac{(p_i^{a_i+1} - 1)}{(p_i - 1)} \quad \text{where} \quad N = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_n^{a_n}$$

represents the prime number breakup of the integer N . One need not necessarily require that this product be actually evaluated. The simplest way to determine a number fraction such as $f(12)$, is to simply write $f(12)=(2+3+4+6)/12=5/4=1.25$. Very often one will find that the sigma function is contained within a computer program such as in MAPLE or MATHEMATICA, and so the evaluation of $f(N)$ for smaller numbers N less than about 40 digits is straight forward. For $N=554400$ we find $f(N)=3.396361833..$. This happens to be the largest $f(N)$ present in the range $1 < N < 600000$. Also our PC can show that-

$$f(1956478234589421383475091235472873409713)= 0.5238100215...$$

and-

$$f(12345678910111213141516171819202122232425)= 0.2400802015...$$

Several interesting general features of $f(N)$ are observed. First of all, whenever a number N is prime, the Number Fraction must necessarily be zero. This condition was built into our original definition of $f(N)$. Any value of $f(N)$ other than zero indicates a composite number with the degree of compositeness increasing with increasing values of $f(N)$. Semi-primes $N=p_1 \cdot p_2$ have their $f(N)$ near zero and (what I term) large or super composites have $f(N)>1$. The classical definition of a perfect number N is one where the sum of all its divisors except N just equals N . That is $\sigma(N)=2N$. In terms of $f(N)$, perfect numbers are given by-

$$f(N) \equiv 1 - \frac{1}{N}$$

The first few perfect numbers found are $N=6, 28, 496$, and 8128 . The $f(N)$ s for these numbers lie slightly below unity. In the literature one also finds references to deficient and abundant numbers. In terms of $f(N)$ these have the definitions-

$$f(N) = \begin{cases} > (1 - \frac{1}{N}) = \text{abundant} \\ < (1 - \frac{1}{N}) = \text{deficient} \end{cases}$$

So for larger N, a number with $f(N) > 1$ is abundant and one where $f(N) < 1$ is deficient. N=12 is an abundant number but N=14 is deficient. I will refer to those Ns where $f(N) > 1$ as large or super-composites. N=1260 , 9240 , 277200, and 3326242920 are all examples of super-composites.

It is possible to work out analytical expressions for certain functional forms of N. Take the case of $N=2^n$ with $n=1, 2, 3$, etc. Substituting into the definition of $f(N)$ we find-

$$f(2^n) = 1 - \frac{1}{2^{n-1}}$$

This leads to $f(N)$ s of intermediate value near one. For Mersenne Primes, defined as $N=2^p-1$ for certain primes p , the value of $f(N)$ is always zero . Another result is-

$$f(3^n) = \frac{1}{2} \left\{ 1 - \frac{1}{3^{n-1}} \right\}$$

which allows us to generalize to-

$$f(p^n) = \frac{1}{(p-1)} \left\{ 1 - \frac{1}{p^{n-1}} \right\}$$

for any prime number p . Note that this result will not work when p is a composite. It also follows that –

$$f(p^2) = \frac{1}{p} \quad \text{and} \quad f(p^4) = \frac{(p^2 + p + 1)}{p^3}$$

From this one can further generalize things to obtain the identity-

$$\sum_{k=1}^{2n-1} \frac{1}{p^k} = f(p^{2n})$$

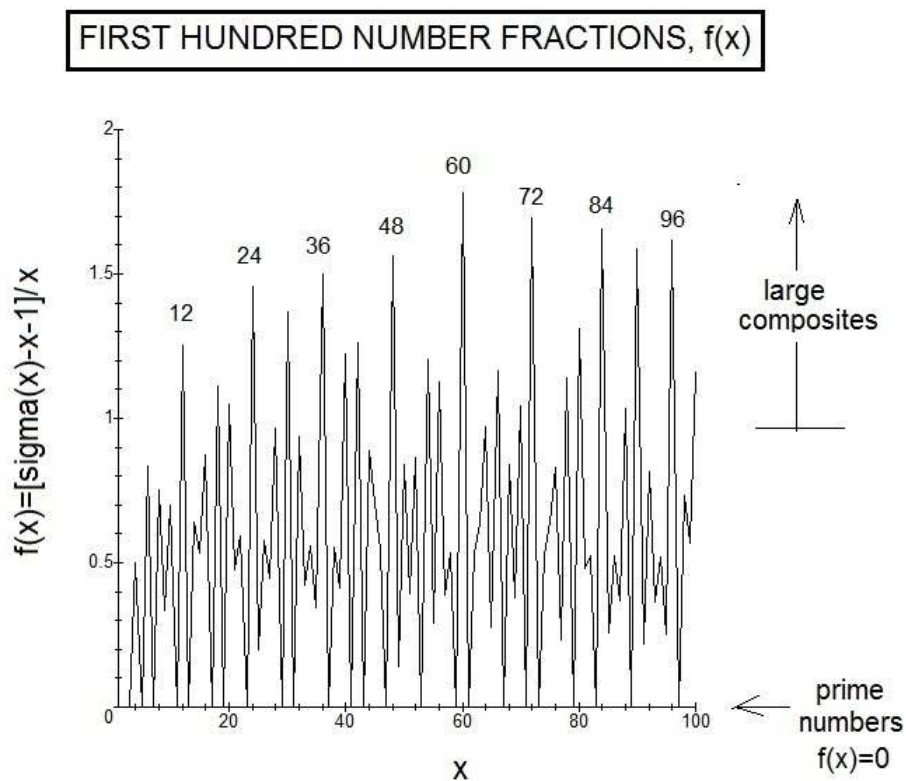
We can also use the $f(p^n)$ result and work backwards to show that the sigma function $\sigma(p^n)$ equals $(p^{n+1}-1)/(p-1)$. This provides an alternate proof for the basic product representation of $\sigma(N)$ given earlier. In addition we have in the limit as $n \rightarrow 1$ that $f(p)=0$.

The simplest way to find super-composites is to carry out the following MAPLE computer scheme-

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with(numtheory): with(plots):
listplot([seq(-c+((sigma(x)-x-1)/x)],x=a..b)],view=[a..b, 0..d]);
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where $a < x < b$ gives the range of x over which we are plotting the number fractions $f(x)$, c is a number greater than 1 for super-primes and d lies between 0 and 3.

For a graph of the first one hundred Number Fractions we use $a=1$, $b=100$, $c=0$ and $d=2$. It produces the listplot-



Notice in this range of x the largest composite is located at $x=60$ and has a number fraction of $f(60)=1.78333...$ To determine the largest values of $f(x)$ in a given range, we typically choose a c greater than about 1. This avoids all the hash appearing for smaller $f(x)$ yet clearly points out the location of super-composites. We have carried out such calculations and find the following results for the largest $f(x)$ s in the ranges indicated-

Range $a < x < b$	Value of x at max $f(x)$	Number Fraction, $f(x)$
$1 < x < 100$	60	1.7833333

$100 < x < 1000$	840	2.4273809
$1000 < x < 10000$	5040	2.8378968
$10000 < x < 100000$	55440	3.1869949

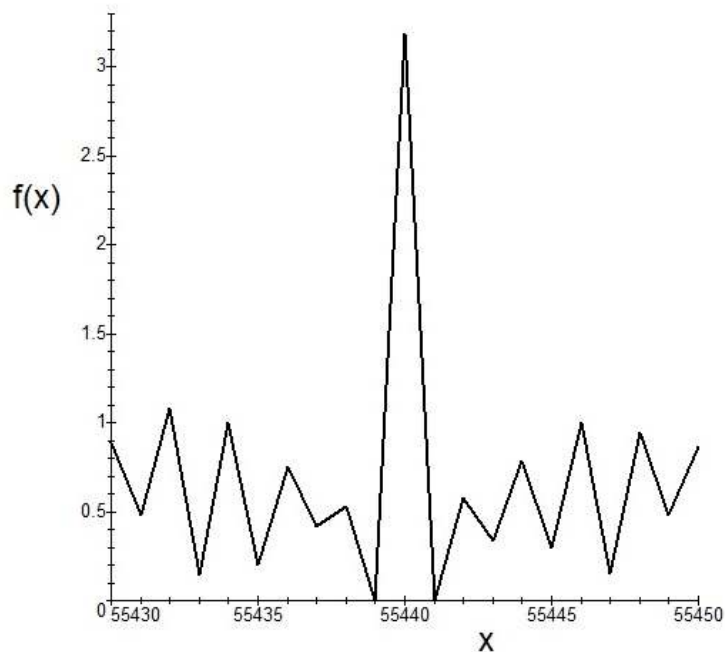
It is seen that the value of the largest super-composite seems to increase very slowly with increasing x but spot checks of $\lim_{n \rightarrow \infty} f(N^n)$ indicate that $f(x)$ is probably bounded. Certain regularities are observed including –

$$\lim_{n \rightarrow \infty} \{ f[(2^k)^n] \} = 1, \text{ and } \lim_{n \rightarrow \infty} \{ f[(6k)^n] \} = 2$$

which hold for most positive integers k . A spot check with the twelve digit number $N = 113210697600$ yields the Number Fraction $f(N) = 4.335823268$. Here $N-1 = (1088533)(104003)$ is a semi-prime while $N+1 = 113210697601$ is a prime number. Note that for all the x s considered above, the value of x at a local max $f(x)$ is divisible by 6.

We can view the detailed structure of x versus $f(x)$ by plotting a narrow range about one of these maxima. Take the case of $x = 55440$ and its ± 10 neighborhood. The curve looks like this-

SUPER-COMPOSITE AT $X = 55440$ WHERE $F(X) = 3.1869949..$
(Note the twin primes at 55439 and 55441)



Note that an approximate mirror reflection occurs about the super-prime at $x=55440$ and also observe the appearance of a double prime at $x=55440\pm 1$. The spike in $f(x)$ is very reminiscent of a rogue wave occurring in a turbulent sea. The observation that primes (or possibly semi-primes) are likely to occur in the immediate neighborhood of a large or super-prime has already been noted by us in an earlier article including the observation that all primes above $p=3$ can be represented as Q primes defined as $Q=6n\pm 1$. It seems that it is statistically likely that the number in the immediate neighborhood of a large or super-composite is likely to be a prime or semi-prime number, although this will become less likely as N becomes very large since the spacing between primes increases with increasing N . As an example, take the case of the super-composite $x=665280=3080\cdot 6^3$ which has $f(x)=3.398266895$. It suggests that either $x+1$ or $x-1$ or both may be prime or semi-prime. Testing things out we find-

$$\text{ifactor}(665281)=577\cdot 1153 \quad \text{and} \quad \text{ifactor}(665279)=665279$$

so that 665279 is indeed a prime number while 665281 is a semi-prime composed of the primes 577 and 1153. The value of $f(665281)=0.0026004049..$ and so is not quite zero. We also observe that large local peaks in $f(x)$ are generated by $x=3080\cdot 6^k$, with k a positive integer. This suggests there will be many primes of the form $x=3080\cdot 6^k\pm 1$. Among those primes are-

$$x:=3080\cdot 6^{11}-1=1117414932479$$

$$x:=3080\cdot 6^{43}-1=8893126766801959400651827451296481279$$

$$x:=3080\cdot 6^{117}+1=34060185694579345976281473557832638374790008390378949646007197472348848463746299874908181626881 \quad \text{and}$$

$$x:=3080\cdot 6^{152}+1=58551870661560298684797939364842668487632579032451525970270599107957711517856662244784711592670849738572021498983093370881$$

We can also generate very large primes by finding that number n where-

$$N=6(\text{Random number}+n)\pm 1$$

is prime. Random numbers can be generated by multiplying various irrational numbers together or just generating them directly with a random number generator. As an example, we looked at the number-

$$N=60((2578398250129356753468308953331+n)-1$$

and found it becomes prime when $n=31$. That is-

$$N=154703895007761405208098537201719$$

is a 33 digit long prime number. We verify this fact by noting $f(N) = [\sigma(N) - N - 1]/N = 0$ as expected.

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