PROPERTIES OF THE FUNCTION $F(x,N)=[1-x^{(N+1)}]/[1-x]$

If we write out the N+1 term series-

$$F(x,N)=1+x+x^2+x^3+\ldots+x^N$$

and then subtract xF(x,N) from it , we get-

$$F(x,N)=[1-x^{(N+1)}]/[1-x]$$

So regardless of the size of x, the finite length series F(x,N) can be represented as a quotient involving x and N. We want in this article to examine the properties of this identity-

$$F(x,N)=1+x+x^2+x^3+\ldots+x^N =[1-x^{(N+1)}]/[1-x]$$

for various forms of x.

One of the simplest forms of F(x,N) occurs when N=infinity and |x|<1. Here we have the classical geometric series-

$$\sum_{n=0}^{\infty} x^n = 1/(1-x)$$

So 1+1/2+1/4+1/8+…=2. Also, on differentiating once, we have-

$$\sum_{n=1}^{\infty} nx\hat{}(n-1)=1/(1-x)\hat{}2$$

At x=1/2, this produces-

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{1}+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\ldots = 4$$

Next going back to the original form of F(x,N) and setting x=cos(θ)^2 and N=infinity, we get-

$$\sec(\theta)=\text{sqrt}[1+\cos(\theta)\hat{}2+\cos(\theta)\hat{}4+\ldots\}$$

At θ=π/4, this produces-

$$\sec(\pi/4)=\text{sqrt}(2)=\text{sqrt}(1+1/2+1/4+\ldots)$$

Note that if N has a finite value then the corresponding series has finite length. An example is for N=4 and x=2. This produces-

$$1+2+4+8+16=(1-2\hat{}5/(1-2)=31$$

Or more generally-

$$\sum_{n=0}^{N} 2\hat{}n = 2^{N+1}-1$$

When N+1 is a prime then 2^p-1 is a Mersenne Number which may or may not be a prime. Of these which are prime we have the following first ten-

2^3-1=7 (prime)

2^5-1=31(prime)

2^7-1=127(prime)

2^13-1=8191(prime)

2^17-1=131071(prime)

2^19-1=524287(prime)

2^21-1=20971519(prime)

2^31-1=2147483647(prime)

2^61-1=2305843009213693951 (prime)

2^89-1=618970019642690137449562111 (prime)

In both cases where 2^N-1 is prime or composite, the following holds-

$$2^N=1+\sum_{n=0}^{N-1} 2^n$$

It is also possible to integrate the function F(x,N) with respect to z. This yields,upon setting N=infinity, the result-

ln(1-x)=-{x+x^2/2+x^3/3+x^4/4+…}

At x=1/2, this leads to the identity-

$$\ln(2)=1/2+1/8+1/24+1/64+\ldots=\sum_{n=1}^{\infty}\frac{1}{n(2^n)}$$

Finally, let us consider the complex version of the geometric series. Letting z=rexpiθ with |z|<1, It reads-

$$\sum_{n=0}^{\infty}[rexp(i\theta)]^n=\frac{1}{1-rexp(i\theta)}$$

This produces the real part-

$$\sum_{n=0}^{\infty} r^n\cos(n\theta)=\frac{1-rco\ \ (\theta}{1-2r\cos(\theta)+r^2}$$

and the imaginary part-

$$\sum_{n=0}^{\infty} r^n\sin(n\theta)=\frac{r\sin(\theta)}{1-2r\cos(\theta)+r^2}\qquad .$$

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