PROPERTIES OF ARCTAN(Z)

We know from elementary calculus that the function z=tan(θ) has an inverse θ=arctan(z). In differentiating z once we have-

$$dz = [1+\tan(\theta)^2]\, d\theta \quad \textit{or its equivalent} \quad \arctan(z) = \int_0^z \frac{dz}{1+z^2}$$

On setting the upper limit to 1/N with N<1 we find the infinite series expansion for arctan given by-

$$\arctan(1/N) = \int_{z=0}^{1/N} \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n}\, dz = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)N^{2n+1}}$$

or the equivalent-

$$\arctan(1/N) = \sum_{n=0}^{m-1} \frac{(-1)^n}{(2n+1)N^{2n+1}} + \int_{t=0}^{1/N} \frac{t^{2m}}{1+t^2} dt$$

This series will converge quite rapidly when N>>1. Thus-

$$\arctan(1/239) = \frac{1}{239}\left[1 - \frac{1}{3(239)^2} + \frac{1}{5(239)^4} - \frac{1}{7(239)^6} + \ldots\right]$$

However for N=1, the series just equals that of Gregory which is known to be notoriously slowly convergent-

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4}$$

If one takes the first hundred terms(m=100) in the Gregory series, the integral remainder will still be-

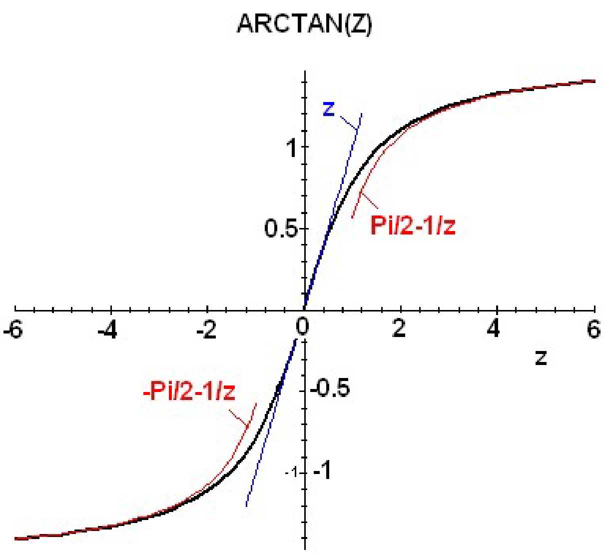
$$\int_{t=0}^{1} \frac{t^{200}}{1+t^2} dt = 0.0024999\ldots \quad \textit{or some} \quad 1/3 \quad \textit{percent}$$

In general the larger N becomes the more rapidly the infinite series for arctan(z) will converge. Thus the series for (π/8) =arctan{ 1/[1+sqrt(2)]} reads -

$$\frac{\pi}{8}=\frac{1}{(1+\sqrt{2})}\left[1-\frac{1}{(1+\sqrt{2})^2}+\frac{1}{(1+\sqrt{2})^4}-...\right]$$

which converges somewhat faster than the Gregory series.

Lets examine some of the other analytical characteristics of arctan(z). Its plot for z real looks like this-



We see that arctan(z) varies linearly with z for small z starting with value zero and becomes non-linear in its variation with increasing z, eventually approaching Pi/2 as Pi/2-1/z as z approaches infinity. The function has odd symmetry since arctan(-z)=-arctan(z). Its derivative is just 1/(1+z^2) and hence represents a special case of the Witch of Agnesi (this curve was studied by the Italian mathematician Maria Agnesi 1718-1799 and received its name due to a mistranslation of the Italian word versiero for curve by an English translator who mixed it up with the Italian word for witch). Using the multiple angle formula for tangent , one also has-

$$\tan[\arctan(x)+\arctan(y)]=\frac{\tan[\arctan(x)]+\tan[\arctan(y)]}{1-\tan[\arctan(x)]\tan[\arctan(y)]}$$

or the equivalent -

$$\arctan[\frac{(x+y)}{(1-xy)}]=\arctan(x)+\arctan(y)$$

On setting x=z and y=∞ we find –

$$\frac{\pi}{2}=\arctan(z)+\arctan(\frac{1}{z})$$

so that, for example, arctan(2)=π/2-arctan(0.5)=π/2-0.46364..= 1.1071…If x=1 and y=-1/3 one obtains the well known identity-

$$\frac{\pi}{4}=\arctan(\frac{1}{2})+\arctan(\frac{1}{3})$$

and x=1/7, y=-1/8 produces-

$$\arctan(\frac{1}{7})=\arctan(\frac{1}{8})+\arctan(\frac{1}{57})$$

Consider next the complex number z=x+iy. Writing this out in polar form yields-

$$x+iy=\sqrt{x^2+y^2}\,\exp[i\arctan(\frac{y}{x})]$$

so that-

$$\arctan(y/x)=-i\ln\left[\frac{x+iy}{\sqrt{x^2+y^2}}\right]$$

This result relates the arctan to the logarithm function so that-

$$\ln\left[\frac{1+i}{\sqrt{2}}\right]=i\frac{\pi}{4}$$

Looking at the near linear relation between arctan(z) and z for z<<1 suggests that arctan(1/N)=m*arctan(1/(m*N) +small correction of the order 1/N^3 for large N. This is indeed the case. By looking at the imaginary part of-

$$\ln\left[(N_1+i)^{p_1}(N_2+i)^{p_2}\right] = p_1\ln(N_1+i) + p_2\ln(N_2+i)$$

one finds-

$$\arctan(\frac{1}{N}) = 2\arctan(\frac{1}{2N}) - \arctan(\frac{1}{N(4N^2+3)})$$

,

$$\arctan(\frac{1}{N}) = 3\arctan(\frac{1}{3N}) - \arctan(\frac{8N}{27N^4+18N^2-1})$$

and-

$$\arctan(\frac{1}{N}) = 4\arctan(\frac{1}{4N}) - \arctan(\frac{80N^2-1}{N(256N^4-160N^2-15)})$$

We next solve an integral in terms of arctan to get-

$$\int\frac{dt}{at^2+bt+c} = \frac{1}{a}\int\frac{dt}{\left[(t+\frac{b}{2a})^2+(\frac{4ac-b^2}{4a^2})\right]} = \frac{2}{\sqrt{4ac-b^2}}\arctan\left[\frac{2at+b}{\sqrt{4ac-b^2}}\right]$$

Therefore one finds-

$$\int_{t=0}^{1}\frac{dt}{t^2+3t+4} = \frac{2}{\sqrt{7}}[\arctan(\frac{5}{\sqrt{7}}) - \arctan(\frac{3}{\sqrt{7}})] = \frac{2}{\sqrt{7}}\arctan(\frac{\sqrt{7}}{11})$$

It is also possible to manipulate the original integral form for artctan(z) into a variety of different range integrals. Consider the substitutions t=u/N and Nt=tanh(v). These produce the integrals-

$$\arctan(\frac{1}{N}) = \int_{t=0}^{1/N} \frac{1}{1+t^2} dt = N \int_{u=0}^{1} \frac{du}{N^2+u^2} = \left(\frac{N}{N^2+1}\right) \int_{v=0}^{\infty} \frac{dv}{[\cosh(v)^2 - (\frac{1}{N^2+1})]}$$

Expanding the term in the denominator of the last integral leads to an alternate series for arctan(1/N). In compact form, it reads-

$$\arctan(\frac{1}{N}) = \left[\frac{N}{N^2+1}\right] \sum_{n=0}^{\infty} \frac{4^n n!^2}{(2n+1)!(N^2+1)^n}$$

and produces the identity-

$$\frac{\pi}{2} = 1 + \frac{1!^2 2^1}{3!} + \frac{2!^2 2^2}{5!} + \frac{3!^2 2^3}{7!} + \frac{4!^2 2^4}{9!} + \ldots$$

Also using the variable substitution u=w/sqrt(w^2+1) yields the symmetric form-

$$\arctan(1/N) = \frac{N}{2(N^2+1)} \int_{w=-\infty}^{w=+\infty} \frac{dw}{\sqrt{(w^2+1)}\,[(w^2+1) - \frac{1}{N^2+1}]}$$

so that-

$$\pi = \int_{-\infty}^{+\infty} \frac{dw}{(w^2+0.5)\sqrt{w^2+1}} = \sqrt{2}\left[\int_{v=0}^{\infty} \frac{dv}{(\cosh(v) - \frac{1}{\sqrt{2}})} - \int_{v=0}^{\infty} \frac{dv}{(\cosh(v) + \frac{1}{\sqrt{2}}}\right]$$

It seems that this last integral in w can form the starting point for an AGM approach for finding precise values of π. It can also be expanded as the series-

$$\frac{\pi}{2} = \frac{0!}{1} + \frac{1!}{1\cdot 3} + \frac{2!}{1\cdot 3\cdot 5} + \frac{3!}{1\cdot 3\cdot 5\cdot 7} + \frac{4!}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \ldots$$

which shows an interesting pattern but is unfortunately only slowly convergent. A much more rapidly convergent series is found for larger N. Indeed, we have in general that-

$$\arctan(1/N) = \frac{N}{N^2+1}\int_0^\infty \frac{dw}{(w^2+1)^{3/2}}\left[\frac{1}{1-\dfrac{1}{(N^2+1)(w^2+1)}}\right]$$

$$= \sum_{n=0}^\infty \frac{N}{(N^2+1)^{1+n}}\int_{w=0}^\infty \frac{dw}{(w^2+1)^{3/2+n}} = \sum_{n=0}^\infty \frac{2^n n! N}{(N^2+1)^{n+1}[1\cdot 3\cdot 5\cdot ...(2n+1)]}$$

This yields at N=239 the result-

$$\arctan(1/239) = \sum_{n=0}^\infty \frac{239(n!)(n+1)!2^{2n+1}}{(2n+2)!(239^2+1)^{n+1}} =$$

$$= .041840760020747238645382149592854527410480 65...$$

which is accurate to 43 places when adding up just the first nine terms in the infinite series. By telescoping the arctan(1/N) series terms by two, one finds the even faster convergent form-

$$\arctan(1/N) = \sum_{n=0}^\infty \frac{4^n N\, n!\Gamma(n+\frac{1}{2})}{(N^2+1)^{2n+2}\Gamma(2n+\frac{1}{2})}\left[\frac{(4n+3)N^2+(8n+5)}{(16n^2+16n+3)}\right]$$

Also using our earlier discussed four term arctan formula for π (see-NUMERICAL EVALUATION OF PI BY A FOUR TERM ARCTAN FORMULA) we have that-

$$\pi = \frac{1}{4}\sum_{n=1}^\infty \frac{4^n n!(n-1)!}{(2n)!}\left[\frac{3648}{(1445)^n}+\frac{9120}{(3250)^n}+\frac{13384}{(57122)^n}+\frac{51456}{(71825)^n}\right]$$

$$= 3.1415926535897932384626433832795028...$$

with each additional term taken in this series improving the accuracy of π by about 3 places. Note the summation procedure requires no taking of roots and simply involves summation, multiplication, and division of integers.

Arctan(z) also relates to the hypergeometric series. Matching term by term of the infinite series for F(a,b,c,x) with the the first infinite series expansion for arctan given earlier, one has-

$$\arctan(z) = zF(1/2, 1; 3/2; -z^2) = \frac{z}{2}\int_{t=0}^{1} \frac{dt}{\sqrt{(1-t)(1+z^2 t)}}$$

Also it follows that the second order differential equation-

$$2z(1-z)\frac{d^2w}{dz^2} + (3-5z)\frac{dw}{dz} - w = 0$$

has-

$$w = \frac{1}{z}\arctan(z)$$

as a solution. Finally, one integration of arctan yields-

$$\int \arctan(z) = z\arctan(z) - \frac{1}{2}\ln(1+z^2)$$

which is easily verified by differentiating both sides.

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