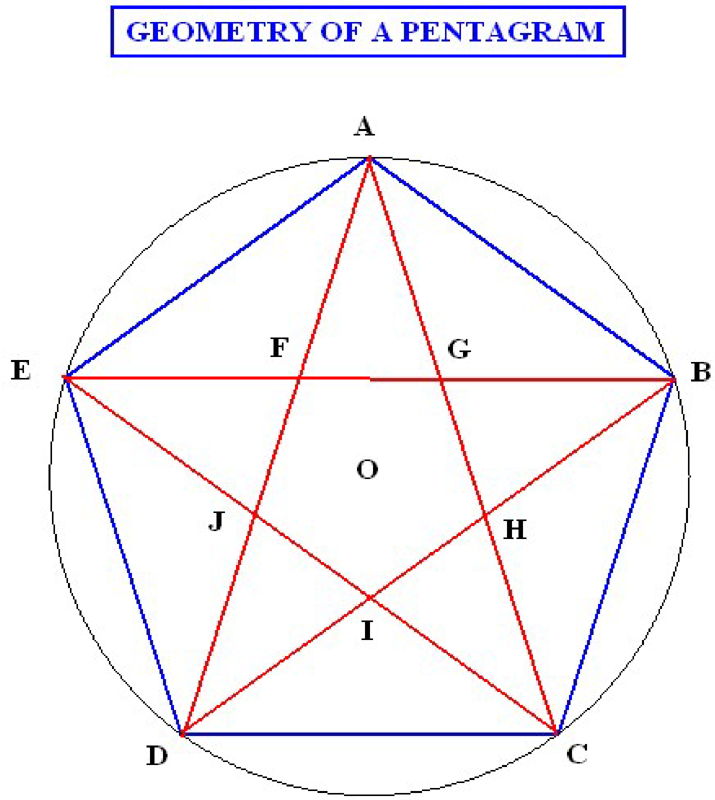
GEOMETRIC PROPERTIES OF PENTAGRAMS

A pentagram is a stellated pentagon in the form of a five pointed star as shown with red borders in the following figure-



It is a symbol whose history goes back to the Babylonians and ancient Greeks and during the early Renaissance became associated with the occult and free-masonry. When the star is pointed up (as in the above diagram) it represents the sea of wisdom, while its flipped form is associated with devil worship. We want here to discuss in detail its geometric properties. Its construction is accomplished by first drawing a circle with radius-

$$R = \sqrt{\frac{1-\cos(\frac{3\pi}{5})}{1-\cos(\frac{2\pi}{5})}} = 1.3763819...$$

The choice for this particular radius is dictated by having the other length dimensions of the figure have relatively simple values. Next one marks off five points A, B, C, D, and E on the circle at angle increments of 2π/5 rad=72deg as measured from the circle origin at O. Then one draws five straight lines from A to C, from C to E, from E to B, from B to D, and from D to A. The resultant figure is a regular pentagram. Connecting the tips of the resultant five point star produces a standard pentagon, and the five sided overlap region near the circle origin represents a smaller flipped pentagon. To measure the various lengths of the straight line segments one starts by noting that the three basic angles are–

$$Angle\ AOB = \frac{2\pi}{5} \quad , \quad Angle\ AGB = \frac{3\pi}{5} \quad and \quad Angle FAG = \frac{\pi}{5}$$

Using the law of cosines one next finds that the lengths are-

$$AB = R\sqrt{2[1-\cos(\frac{2\pi}{5})]} \quad and \quad AG = R\sqrt{\frac{[1-\cos(\frac{2\pi}{5})]}{[1-\cos(\frac{3\pi}{5})]}} = 1$$

If one now takes the ratio of AB to AG, we find-

$$\frac{AB}{AG} = \sqrt{2[1-\cos(\frac{3\pi}{5})]} = 2\sin(\frac{3\pi}{10}) = 1.680339..$$

This ratio is recognized as the Golden Ratio-

$$\Phi = \frac{[1+\sqrt{5}]}{2} = \mathbf{1.61803398874989\ldots}$$

with the property that –

$$\Phi^2 = \frac{[3+\sqrt{5}]}{2} = 1+\Phi, \quad \Phi^3 = 2\Phi+1, \, and\ \Phi^n = F_{n-1}\Phi + F_{n-2}$$

where F_n are the Fibonacci Numbers 1, 2, 3, 5, 8, 13,.. The length AB is thus Φ. Also we have that-

$$\frac{FG}{AG} = 2\cos(\frac{2\pi}{5}) = 2-(AB)^2 = 2-\Phi^2 \ or\ FG = \frac{1}{\Phi}$$

Thus we have the interesting property of the pentagram that-

$$AB:AG:FG=\Phi:1:\frac{1}{\Phi}$$

In terms of the Golden Ratio the circle radius becomes-

$$R=\sqrt{\frac{2\Phi+1}{2\Phi-1}}$$

and the pentagram circumference C=10(AG) =10 . The pentagram area is-

$$Area=\frac{5}{2}\Phi\left[R\cos(\frac{\pi}{5})-\cos(\frac{3\pi}{10})\right]=\frac{5}{4}\Phi\left[\sqrt{\frac{5\Phi+3}{2\Phi-1}}-\sqrt{3-\Phi}\right]$$

This evaluates to Area=2.126627… meaning that the fraction of the circle area covered by the pentagram is-

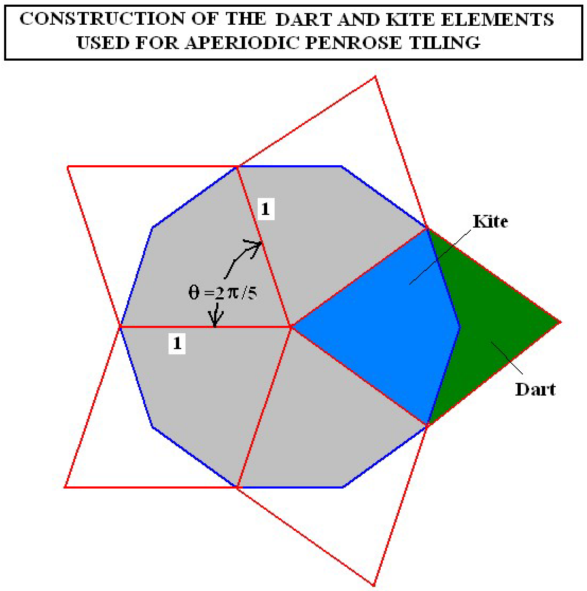
$$f=2.126627\ldots/[\pi(1.3763819)^2]=0.3573\ldots$$

Since the area of a regular pentagon is known to be $A_{pentagon}=(5s^2/4)\cot(\pi/5)$ with s the side length, we find the outer and inner pentagons in the above figure to have areas-

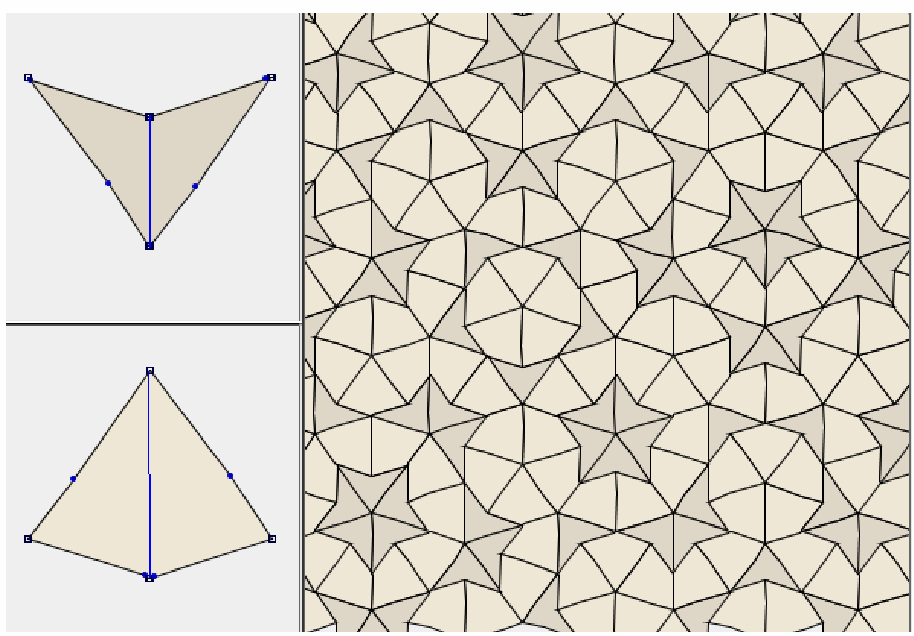
$$A_{pout}=\frac{5}{4}\Phi^2\cot(\frac{\pi}{5})=\frac{5}{4}\left[\frac{1+2\Phi}{\sqrt{3-\Phi}}\right]\quad and\quad A_{pin}=\frac{5}{4\Phi}\left[\frac{1}{\sqrt{3-\Phi}}\right]$$

You will note that the area ratio of the outer to inner pentagon is just $3\Phi+2=\Phi^4$ since the length ratio of the sides s is just Φ^2.

Another interesting property of pentagrams is that their sub areas may be used to produce an aperiodic tiling (ie. Penrose Tiling) covering all points in a plane. The sub-area used in the tiling are easiest to construct by drawing a decagon and then adding five rhombus areas as shown-



The two shapes (termed kite and dart) constitute the basis for Penrose tiling and also follow from our original pentagram figure given above by use of the triangles ABG and ABH. We show you here an internet published version of the non-periodic tile pattern emerging when using these two elements-



source- http://www.cgl.uwaterloo.ca/~csk/software/penrose/

Notice there are no gaps or overlaps between the tiles. Such aperiodic tilings are not quite as aesthetically pleasing as the very intricate periodic tilings one finds at the Alhambra palace in Granada, Spain or the Topkapi palace in Istanbul, Turkey.

January 2011