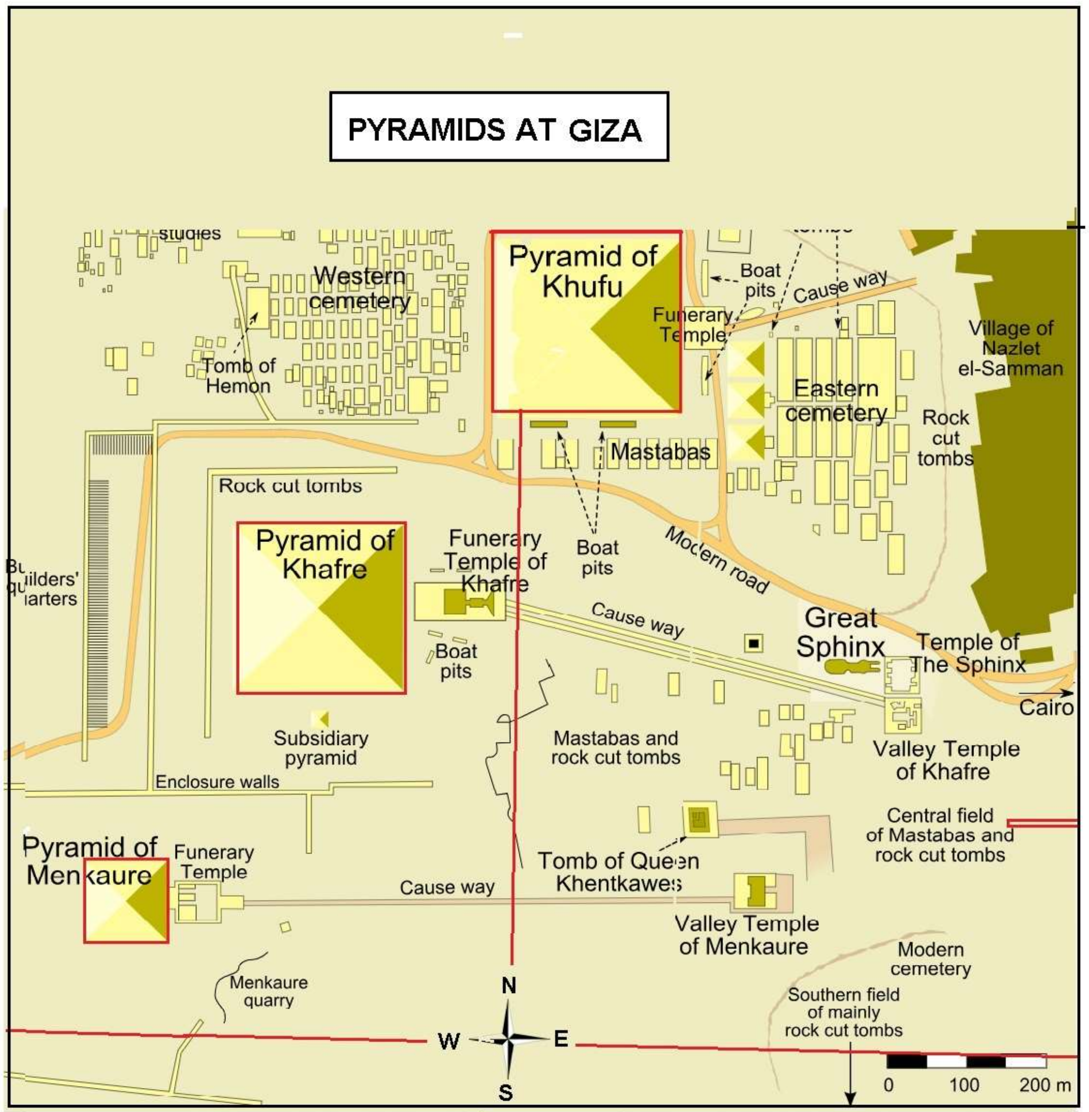
COMPARING THE GREAT PYRAMID OF KHUFU WITH THAT OF ANY REGULAR SQUARE BASE PYRAMID

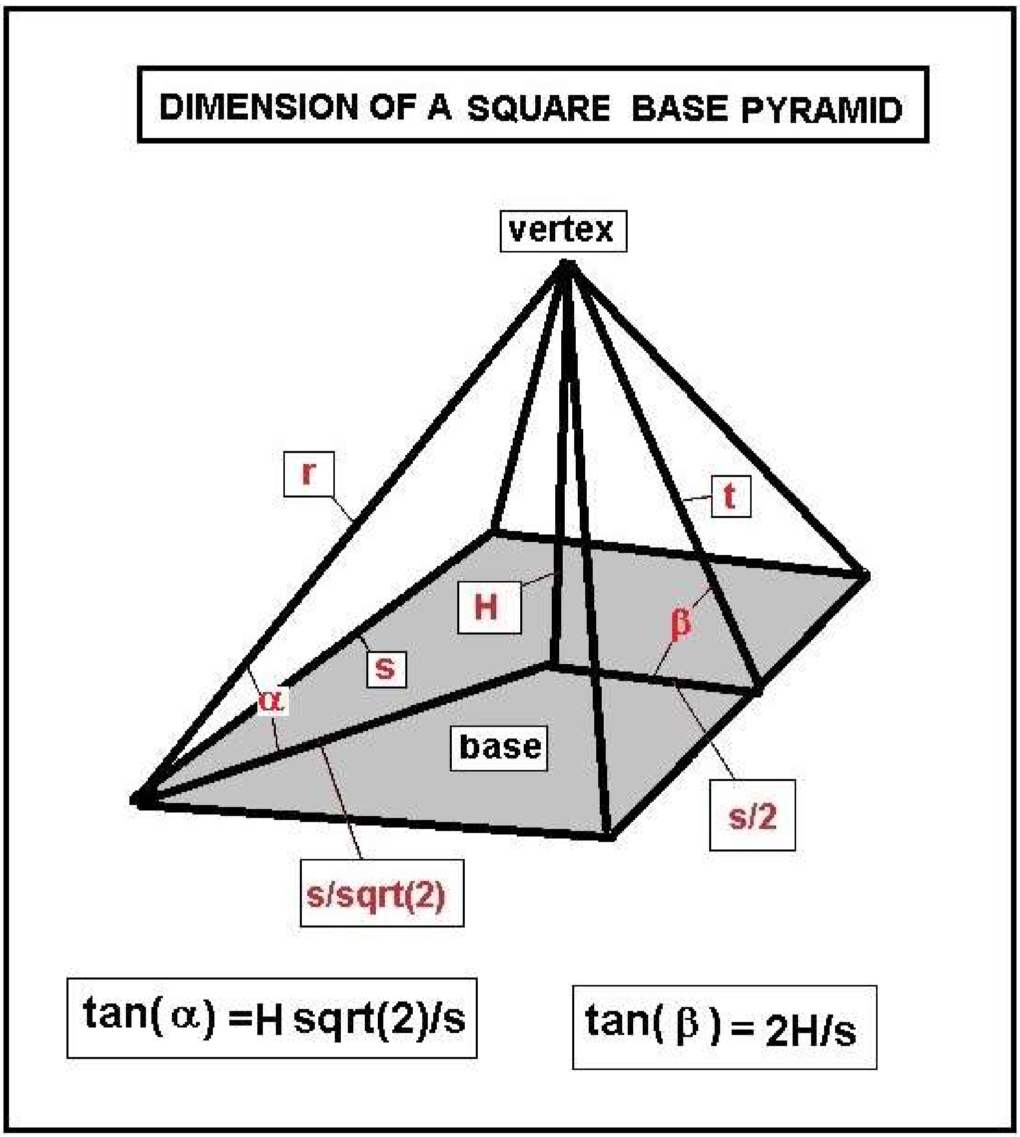
The zenith of Egyptian Pyramid building was reached about 4500 years ago by the three square base pyramids of Khufu, Khafre, and Menkaure located on the Giza plateau just outside of present day Cairo. Here is a sketch of their orientation showing a nearly perfect alignment with a north-south line-



How such an alinement was reached by the ancient Egyptians is still somewhat of a mystery. Most likely they used the shadow cast by a pelekinon (vertical pole) or used the location of their north star at that time.

We want in this note to compare the dimensions of a classic square base pyramid with that of Khufu's Pyramid (also referred to as the Great or Cheops Pyramid) to see how close this wonder of the ancient world comes to a true square base pyramid with equilateral sides

We begin by looking at the schematic of a classic square base pyramid. It has four triangular sides plus a square base as shown-



It has a total of five faces F, five verteces V, and eight edges E. As expected it satisfies the Euler's Formula –

$$F+V-E=2$$

We also have from Pythagoras that-

r=sqrt[H^2+s^2/2]   and   t=sqrt[H^2+s^2/4]

, where s^2 represents the base area and H is the pyramid height.The two angles have tangents as shown in the schematic. Note that the tangent ratio reads-

tan(□□)/ tan(□□)=sqrt(2)

and so is independent of the pyramid height.

The volume of this pyramid equals-

$$V=s^2\int_{z=0}^{H}\left(1-\frac{z}{H}\right)^2 dz = Hs^2/3$$

For the Eqptian type of square base pyramid one requires that r=s in order to insure that the four side surfaces are all equilateral triangles. Under that restriction the volume simplifies to-

V=s^3/[3 sqrt(2)] since His now s /sqrt(2)

A less attractive square pyramid, having H>s/sqrt(2), is found on the back of the US dollar bill.

The tangents for an Egyptian square base pyramid are-

tan(□)=1   and   tan(□)**=sqrt(2)**

These produce –

□□□□□□□□□□□□□□□□□□□□□□□deg□□□□□□□and
□□□□□□□□□□□□deg

for the ideal Egyptian square base pyramid.

To compare these results with the Pyramid of Khufu, we start with the known measured values–

s=230 m , H=147 m and angle □=51.87 deg

These produce –

□□□□arctan[147sqrt(2)/230]=41.37 deg

So we see that the angle alpha is a little smaller than the 45 deg required for the ideal Egyptian pyramid. The reason for this discrepancy is most likely due to the pyramid builders worry of having angle beta too large which could cause a collapse as occurred at one of their earlier pyramids further up the Nile. It is also possible they where trying to reduce the number of stones required for the pyramid construction.

One of the more remarkable mathematical formulas that the pyramid builders used over four thousand years ago describes the fraction f of stones which have been used on reaching the intermediate height h for a pyramid of height H. The formula reads-

f=(h/H)[3-3(h/H)+(h/H)^2]

Today it is easy to derive this formula by calculus, but no one knows how the pyramid builders found it. Here is a modern day derivation-

$$f=\int_{z=0}^{h} s^2(1-\frac{z}{H})^2/(s^2H/3)dz=(3/H)\int_{z=0}^{h}(1-\frac{z}{H})^2dz$$

=(h/H^3)[3H^2-3Hh+h^2]=(h/H)[3-3(h/H)+(h/H)^2]

We see from this last result that f=1 when h=H and f=0 when h=0. What is most interesting about this cubic equation is that it predicts f=1/2 when h/H is only 0.206299.. . This means that half of the stone stacking has been accomplished when one has reached just twenty percent of the pyramid’s final height. This low center of mass of any square base pyramid also explains why it is so resistant against earthquakes.

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