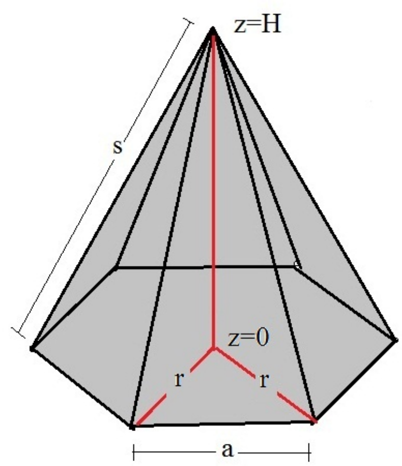
THE GEOMETRY OF PYRAMIDS

One of the more interesting solid structures which has fascinated individuals for thousands of years going all the way back to the ancient Egyptians is the pyramid. It is a structure in which one takes a closed curve in the x-y plane and connects straight lines between every point on this curve and a fixed point P above the centroid of the curve. Classical pyramids such as the structures at Giza have square bases and lateral sides close in form to equilateral triangles. When the closed curve becomes a circle one obtains a cone and this cone becomes a cylindrical rod when point P is moved to infinity. It is our purpose here to discuss the properties of all N sided pyramids including their volume and surface area using only elementary calculus and geometry.

Our starting point will be the following sketch-

SCHEMATIC OF A REGULAR PYRAMID



The base represents a regular N sided polygon with side length 'a' . The angle between neighboring radial lines r (shown in red) connecting the polygon vertices with its centroid is θ=2π/N. From this it follows, by the law of cosines, that the length r=a/sqrt[2(1-cos(θ))] . The area of the iscosolis triangle of sides r-a-r is-

$$A_T = \frac{a}{2}\sqrt{r^2 - \frac{a^2}{4}} = \frac{a^2}{4}\sqrt{\frac{1+\cos(\theta)}{(1-\cos(\theta)}}$$

From this we have that the area of the N sided polygon and hence the pyramid base will be-

$$A_{base} = \frac{Na^2}{4}\sqrt{\frac{1+\cos(\frac{2\pi}{N})}{1-\cos(\frac{2\pi}{N})}}$$

It readily follows from this result that a square base N=4 has area $A_{base=}a^2$ and a hexagon base N=6 yields A_{base}= 3sqrt(3)a^2/2.

Next using the Pythagorean Theorem one has that the slanted edges will each have a length of-

$$s = \sqrt{H^2 + \frac{a^2}{2(1-\cos(\theta))}}$$

To calculate the volume of this pyramid we envision a horizontal cut through the pyramid at height z above the x-y plane. By simple geometry we then have that this produces a smaller N sided polygon of area-

$$A_{polygon} = \frac{Na^2}{4}\left[1-\frac{z}{H}\right]^2\sqrt{\frac{[1+\cos(\theta)]}{[1-\cos(\theta)]}}$$

Applying elementary calculus to this area we get the pyramid volume-

$$V_{pyramid} = const.\int_{z=0}^{z=H}(1-\frac{z}{H})^2 dz = \frac{Na^2H}{12}\sqrt{\frac{1+\cos(\frac{2\pi}{N})}{1-\cos(\frac{2\pi}{N})}} = BaseArea\cdot(\frac{H}{3})$$

For a square base pyramid this result predicts that $V_{pyramid}=a^2H/3$. That is, the volume equals the base area times the pyramid height divided by three. This result continues to hold for all N sided polygon bases including the cone where $N\to\infty$ but $a\to 0$ and the volume becomes $V_{cone}=\pi r^2(H/3)$.

The side surface area of an N sided regular pyramid can be calculated by looking at the individual iscosolis triangles of sides s-a-s, with s being the slant height given earlier. The total surface area produced by these N triangles is-

$$A_{sides} = \frac{aN}{2}\sqrt{H^2 + \frac{a^2}{4}\left[\frac{1+\cos(\theta)}{1-\cos(\theta)}\right]}$$

The simplest pyramid is found when N=3 . It corresponds to a standard tetrahedron when the height H above the x-y plane is set to a×sqrt(2/3). This represents a figure composed of four equal equilateral triangles of side length ‘a’ each. One possible configuration has the four vertices located at [[a/2, a/(2sqrt(3)), 0], [-a/2, a/(2sqrt(3)), 0], [0, -a/sqrt(3), 0], and [0, 0, asqrt(2/3)].The volume of this tetrahedron is $V_{tetrahedron}=a^3/[6sqrt(2)]=0.11785\ a^2$ and the total surface area including the bottom is $A_{tetrahedron}=sqrt(3)a^2=1.73205\ a^2$.

Next let us examine the world’s best known square based pyramid structure, namely, the Great Pyramid of Cheops(alias Khufu) found at Giza, Egypt. This is a most impressive structure built about 2600BC. Its base length is a=775.8 ft and height H=481.4 ft. The four sides are in the shape of iscosolis triangles with the edges having the length of s=719.3 ft. That is, the sides differ slightly from equilateral triangles which would require a height of H=a/sqrt(2)=548.6 ft. Undoubtedly the pyramid builders where worried about pyramid stability and cost by choosing to go for a smaller height than required for a true square base pyramid with equilateral triangle sides. The volume of the Great Pyramid can be calculated using the above given dimensions and is-

$$V_{Cheops} = (1/3)a^2H = \frac{(775.8)^2(481.4)}{3} = 96.579\ \textit{million cubic feet}$$

To be able to cut and move this amount of stone in the estimated 20 year building time for the great pyramid is indeed an amazing feat considering the builders were working without the aid of machines or animals other than the human labor of thousands.

An interesting papyrus found at the Valley of the Kings near Luxor shows that the ancient Egyptians were quite familiar with the properties of square based pyramids. In particular the so-called Moscow Papyrus shows that the builders already knew some 4000 years ago that the volume of a truncated pyramid(frustrum) equals-

$$V_{frustrum} = \frac{h}{3}(a^2 + ab + b^2)$$

,where h is the height of the truncated pyramid, a the base side length , and b the side length at the top level. This formula was derived without the aid of calculus and clearly shows that the architects knew that the volume of any square based pyramid of height H equals $V=a^2H/3$. My guess is that they derived this volume result trough small model experiments using sand to measure volume. Let us derive the Moscow Papyrus result by noting that a truncated pyramid has volume-

$$V = \frac{1}{3}a^2H - \frac{1}{3}b^2(H-h) = \frac{1}{3}a^2H\left\{1-(1-\frac{h}{H})^3\right\}$$

since b/a=1-(h/H). On further expanding we find-

$$V = \frac{h}{3(a-b)}\left\{a^3 - b^3\right\} = \frac{h}{3}\left\{a + ab + b^2\right\}$$

in agreement with the papyrus result. What the builders obviously knew is that the majority of the stone volume is located in the bottom portion of a pyramid. They knew that when h reached half the final height, so that b=a/2, a full 7/8th of the pyramid stones were already in place (ie. $V_{frust}/V_{total}=\{1-(1-1/8)^3\}=7/8$).The Great Pyramid of Cheops has 210 layers of stone, so that when they got to the 105th layer only 1/8th of the stone volume still needed to be installed.

If you ever get a chance to travel to Egypt be sure to visit the pyramids at Giza. These structures are indeed an impressive sight. I remember visiting them and the Sphinx several decades ago. Except for the oppressive outside summer heat and the high humidity inside the pyramids, it was an unforgettable experience. My understanding is that in the last few days the State Department has advised tourist from going there because of lack of police protection for travelers and political instability in Egypt.

Finally it should be pointed out that there are many other pyramidal structures whose bases are not regular polygons and where point P at z=H does not lie directly above the base centroid. All that is required is that one can express the cross-sectional area of a cut through the structure at height z to obtain the pyramid volume via simple calculus. Take, for instance, the volume formed by the slanted plane x+y+z=1 and the x=0,y=0, and z=0 planes. The base area is here ½ and the height is 1. Thus the volume is V=(1/2)(1)/3=1/6. A more complicated structure, reminiscent of some of the Chinese and Japanese pagodas , involves a square base of area a^2 with curved sides merging at a point at z=H. In this case one has a pyramid volume of-

$$V_{pyramid} = a^2 \int_{z=0}^{z=H} \left[1 - \left(\frac{z}{H}\right)^{1/k} \right]^2 dz = \frac{2a^2 H}{(k+1)(k+2)}$$