ROOTS OF INTEGERS USING THE DIOPHANTINE EQUATION y^2=1+(Ax)^2

INTRODUCTION:

If one looks at the non-linear Diophantine Equation (also known as the Brahmagupta Equation) y^2=1+(Ax)^2 , we see that it has integer solutions for certain values of A. Rewriting the equation as a Biharmonic Series, we have the equivalent form-

$$y = Ax\left\{1+\frac{1}{2(Ax)^2}-\frac{1}{8(Ax)^4}+\frac{1}{16(Ax)^6}-\frac{5}{128(Ax)\text{^}8}+\frac{7}{256(Ax)\text{^}10}-\frac{21}{1024(Ax)\text{^}12}+\right\}$$

provided that Ax>>1. Alternatively, we can re-write the equation as the continued fraction-

$$y = Ax+\cfrac{1}{2Ax+\cfrac{1}{2Ax+\cfrac{1}{2Ax+\cfrac{1}{2Ax+}}}}$$

It is our purpose in this note to show how the above expansions lead to some interesting forms for square roots of integers .

We begin by letting A=sqrt(N) and then rewrite the above Binomial Expression as-

$$\text{sqrt(N)=}\left(\frac{Nx}{y}\right)\{1+\frac{1}{1!2(Nx^2)}-\frac{1}{2!2^2(Nx^2)^2}+\frac{1\cdot 3}{3!2^3(Nx^2)\text{^}3}-\frac{1\cdot 3\cdot 5}{4!2^4(Nx^2)^4}+\}$$

This represents a rapidly convergent series for sqrt(N) when the integer solutions [x,y] of the accompanying Diophantine Equation have large values.

SQUARE ROOT OF TWO:

We begin with A=sqrt(2). Here the original Diophantine Equation reads-

y=sqrt(1+2x^2)

The obvious base solution is $[x_0,y_0]$=[0,1] . This is followed by $[x_1,y_1]$=[2,3] and $[x_2,y_2]$=[12,17].Higher integer solutions follow by carrying out the search program-

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for x from a to b do {n,sqrt(1+2x^2)}od;
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Here a and b are chosen by making use of the fact that when x gets large the ratio x_{n+1}/x_n equals 3+2sqrt(2)=5.8284272. A table for $[x_n,y_n]$ going from n=1 through n= 12 follows-

x_n	y_n
2	**3**

12	**17**
70	**99**
408	**577**
2378	**3363**
13860	**19601**
80782	**114243**
470832	**665857**
2744210	**3880899**
15994428	**22619537**
93222358	**131836323**
543339720	**768398401**

Rewriting the above series expansion for A=sqrt(2) using any [x,y] combination in the above table produces-

$$sqrt(2) = (\frac{2x}{y})\{1 + \frac{1}{4x^2} - \frac{1}{32x^4} + \frac{1}{128x^6}\}$$

for a four term Binomial Expansion. Evaluating yields-

sqrt(2)≈ 1.414213562373095048801688724209698078569671875376948073176679737990732478

Thus is accurate to 73 places. The rate of convergence will be less if one takes one of the lower values of [x,y].

SQUARE ROOT OF THREE:

We consider next A=sqrt(3)=1.732050808… . To get a rapidly convergent series for this root, we first construct an [x,y] table using the search routine-

for x from a to b do {x,sqrt(1+3x^2)}od;

This table begins with $[x_0,y_0]$= [1,2] followed by $[x_1,y_1]$=[4,7] and $[x_2,y_2]$=[15,26]. We expect the ratio x_{n+1}/x_n to approach 2+sqrt(3)=3.73205 and y_n/x_n to approach sqrt(3). Carrying out the search we find the following table-

x	y
1	2
4	7
(15	26
56	97
209	362
780	1351
2911	5042
10864	18817
40545	70226

151316	262087
564719	978122
2107560	3650401

Letting x=2107560 and y=3650401, we find taking just the first four terms in the Binomial Expansion , that-

sqrt(3)≈ (3x/y){1+1/(6x^2)-1/(72*x^4)+1/(432*x^6)}=

1.7320508075688772935274463415058723669428052538103806

accurate to the first 53 digits shown.

CONCLUDING REMARKS:

We can also use the continued fraction-

$$y = Ax + \cfrac{1}{2Ax + \cfrac{1}{2Ax + \cfrac{1}{2Ax + \cfrac{1}{2Ax +}}}}$$

, where A=sqrt(3), to get an estimate for this root. Expanding out the first three terms, using the earlier values for x and y, yields the cubic-

4(Ax)^3-4y(Ax)^2+3(Ax)-y=0

Solving we find-

A=sqrt(3)≈ 1.7320508075688772935274463415058723669428

good to 41 places.

We have shown in the above that one can obtain highly accurate approximations to the square roots of any positive integer N using the higher n solutions of a non-linear Diophantine Equation. Detailed calculations have been carried for both sqrt(2) and sqrt(3). In general one has that-

$$\text{sqrt(N)}=\left(\frac{Nx}{y}\right)\{1+\frac{1}{2Nx^2}-\frac{1}{8N^2x\text{^}4}+\frac{1}{16N^3x\text{^}6}-\frac{5}{128N^4x\text{^}8}+\}$$

with [x,y] being higher integer solutions of y=sqrt[1+N(x)^2].

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May 3, 2021
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