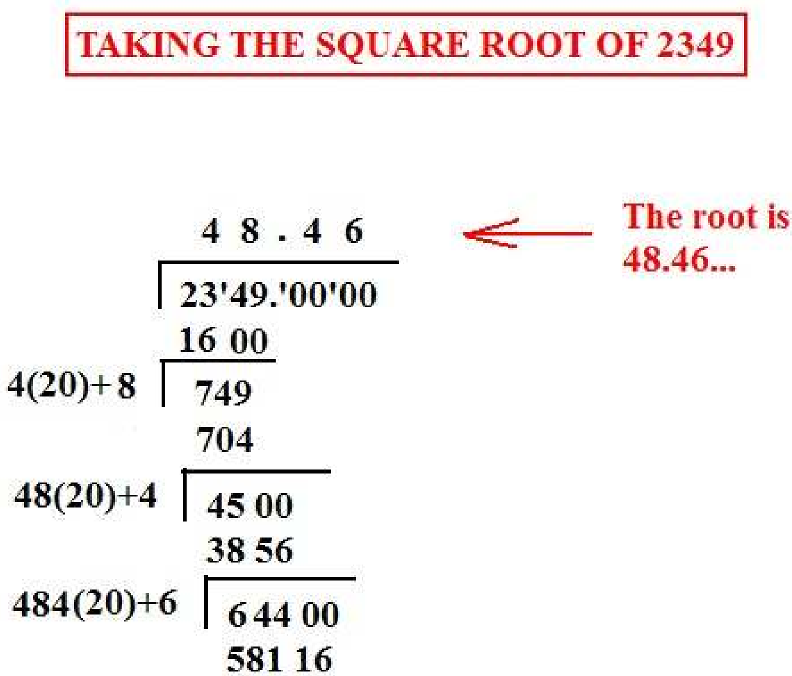
FINDING THE SQUARE ROOT OF NUMBERS

The usual way to calculate the square root of a number these days is to use your hand calculator. When I went to high school over fifty years ago , during a time when there were no PCs, we used either slide rules to find the approximate root of a number or used an iterative technique which works as follows:

Start with a number, say N= 2349. Break it into groups of two starting from the right. Then pick the largest integer root of the pair furthest to the left. In this case it is 4. Next multiply this number by 20 to get 80 and add the number which when added to 80 yields the highest integer divisible into 2349-1600=749. This is the number 8 with a remainder of 749-88(8)=45. Put the 8 next to the earlier found 4. Multiply 48 by 20 to get 960 and add the largest integer allowing division of it plus 960 into 3900 . This integer is 4 since 960+4 divided into 4500 is 4 with a remainder of 4500-4(964)=644. Move the 4 next to the 48 to get the three digit accurate root sqrt(2349)=48.4 Higher accuracy can be obtained by continuing the process further. On a blackboard the process looks like this-



In school we were taught this procedure by rote never really understanding its origin. It is now clear that it is a repeated iteration process. The first step brings out the number 40. The next step the number 8. The third step the number 0.4 and the fourth step the number 0.06. Adding things up yields the answer.

To show in detail how iteration is used in disguised form to produce this answer we start with the equality-

$$(\sqrt{N}-\sqrt{N_0})(\sqrt{N}+\sqrt{N_0})=(N-N_0)$$

with sqrt(N) the sought after root and sqrt(N_0) a known root nearby. We can re-write this equality as-

$$\sqrt{N}=\sqrt{N_0}+\frac{(N-N_0)}{(\sqrt{N_0}-\sqrt{N})}$$

or as the continued fraction-

$$\sqrt{N}=\sqrt{N_0}+\cfrac{(N-N_0)}{2\sqrt{N_0}+\cfrac{(N-N_0)}{2\sqrt{N_0}+\cfrac{(N-N_0}{2\sqrt{N_0}+...}}}$$

If now N=2349 and N_0=1600, we have the identity-

$$\sqrt{2349}=40+\cfrac{749}{2(40)+\cfrac{749}{2(40)+...}}$$

You will see that 2(40) is the factor 20(4) and that 749/80 is more than 8 so if we divide 749/(80+8) we get 8 with a remainder of 45. So we can write-

$$\sqrt{2349}-40-8=\cfrac{4500}{20(48)+\cfrac{4500}{20(48)+..}}$$

so 0.4 is the next term giving us-

$$\sqrt{2349}=40+8+0.4+=48.4...$$

This procedure reproduces exactly the answer gotten by the formerly taught high school procedure.

We can also carry out the direct iteration-

$$S[n+1] = \sqrt{N_0} + \frac{(N - N_0)}{\sqrt{N_0} - S[n]} \quad subject \quad to \quad S[0] = \sqrt{N_0}$$

where the square root of N is given by S[n+1] as n->∞.

Let us demonstrate the iteration for sqrt(3) using N_0=2.89. We have-

$$S[n+1] = 1.70 + \frac{(3 - 2.89)}{1.70 - S[n]} \quad with \quad S[0] = 1.70$$

Which yields the 36 digit accurate result-

Sqrt(3)≈ S[10] = 1.73205080756887729352728313834200813

at the tenth iteration.

One can speed up the iteration (as we first discovered a few days ago) by taking the pth power of sqrt(N) ±sqrt(N_0). This yields-

$$(\sqrt{N} - \sqrt{N_0})^p = \frac{(N - N_0)^p}{(\sqrt{N_0} + \sqrt{N})^p}$$

The term on the left of this expression may be re-writen as a*sqrt(N)-b to produce

the iteration-

$$S[n+1] = \frac{1}{a}\left\{ b - \frac{(N - N_0)^p}{b + aS[n]} \right\} \quad subject \quad to \quad S[0] = \sqrt{N_0}$$

To demonstrate, consider the setting N=5 , N_0=4, and p=4 .This yields the iteration-

$$S[n+1] = \frac{1}{72}\left[161 + \frac{1}{161 + 72S[n]} \right]$$

Starting with S[0]=2 , it produces the 36 digit accurate result-

Sqrt(5)≈ S[7]= 2.23606797749978969640917366873127623

at the seventh iterations . If we let p get even larger then, in view of the above continued fraction form, we have the further approximation-

$$\sqrt{N} \approx \frac{1}{a}\left\{b - \frac{(N-N_0)^p}{2b}\right\}$$

where –

$$(\sqrt{N} - \sqrt{N_0})^p = a\sqrt{N} - b$$

Since the values of a and b are easy to obtain, this root N approximation should yield accurate results when p becomes large. Take the case of N=2, N_0=1, and p=32. Here we find-

b=886731088897 and a=627013566048

so that-

$$\sqrt{2} \approx \frac{1}{627013566048}\left\{886731088897 - \frac{1}{2(886731088897)}\right\}$$

$$= \frac{1572584048032918633353217}{1111984844349868137938112}$$

$$= 1.414213562373095048801688724209698078569671875377$$

This result is accurate to 48 places.

Let as apply this last root approximation to the first number discussed above, namely, N=2349. For N_0 we take the neighboring number 2500 =50^2. With p=8 we find-

$$(\sqrt{2349} - 50)^8 = 4072043185311616 - 756159436368000\sqrt{29}$$

Thus-

$$\sqrt{2349} =$$

$$\frac{9}{756159436368000}\left\{4072043185311616 - \frac{(2349-2500)^8}{2(407204318531161600)}\right\}$$

So that we find the 13 place accurate root –

$$\sqrt{2349} \approx 48.46648326421053$$

Let us finish up by getting the continued fraction for asqrt(N)-b. One has

$$a\sqrt{N} = b - \frac{(N-N_0)^p}{(a\sqrt{N}+b)}$$

This leads to the continued fraction-

$$\sqrt{N} = \frac{1}{a}\left[b - \cfrac{(N-N_0)^p}{2b - \cfrac{(N-N_0)^p}{2b - \cfrac{(N-N_0)^p}{2b - \ldots}}}\right]$$

This continued fraction will converge much faster than the standard continued fraction for roots of numbers, especially when p gets large so that b becomes large. For b >>1, a good approximation for sqrt(N) is simply b/a. This approximation produces the 26 digit accurate result –

sqrt(2)≈ 1.41421356237309504880168

using the values of a and b given above.

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