

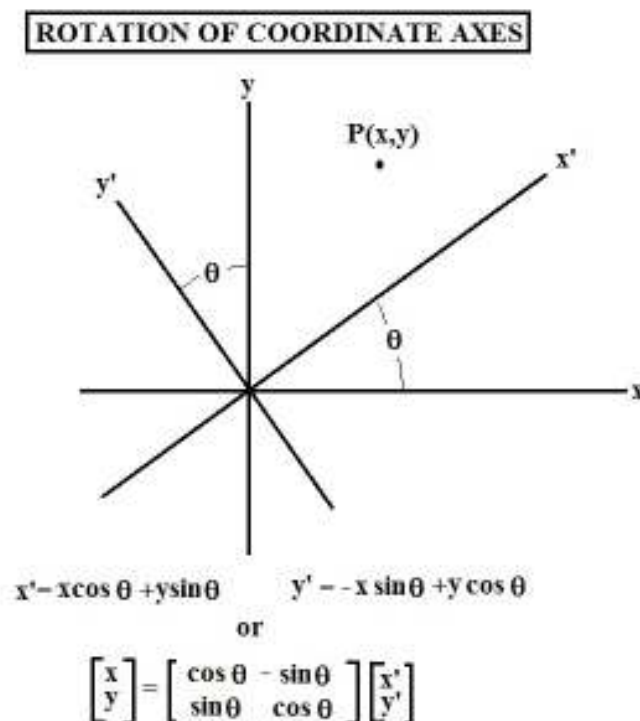
ROTATION OF CURVES IN TWO DIMENSIONS

The ability to rotate figures in both two and three dimension is an important aspect of computer graphics. Although many of us were first introduced to the rotation of certain classical figures such as ellipses and hyperbolas in high school or early college, it was never made quite clear in our analytic geometry class how one accomplishes such rotations via simple matrix manipulations. Rather we were exposed to certain rules such as the appearance of a product term xy in an equation usually implied that one was looking at one of the standard 2D curves under rotation. For example, if one saw a formula such as-

$$17x^2 - 30xy + 17y^2 = 32$$

it was at once suspected that it represents some type of ellipse made clear by substituting the new variables $x' = ax + by$ and $y' = cx + dy$ into the equation and then adjusting the constants a , b , c , and d until the cross term xy disappears. No mention was ever made about the much faster way of using matrix algebra to accomplish this task. We show in this note how one can quickly rotate any figure $F(x,y)=0$ in 2D by any desired angle θ by simple matrix manipulations.

Our starting point is to consider two orthogonal Cartesian coordinate systems $[x,y]$ and $[x',y']$ having the same origin but their axes separated by angle θ from each other. Here is a schematic of the two systems-



The relation between the coordinates is $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$. In matrix form one has-

$$X=MX'$$

where -

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad X' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad \text{the rotation matrix} \quad M = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

One can multiply this matrix equation by its inverse M^{-1} to get the equation-

$$X'=M^{-1}X$$

The two algebraic expressions corresponding to this last equation are given at the bottom of the figure.

Consider now a general quadratic curve defined by-

$$F(x,y)=ax^2+2bxy+cx^2+ex+fy+h=0$$

where a, b, c, e, f and h are specified constants. Special subclasses of this curve are the ellipse, parabola and hyperbola which have the standard forms-

$$(x/a)^2+(y/\beta)^2=1, \quad y=\alpha+\beta x^2, \quad \text{and} \quad (x/a)^2-(y/\beta)^2=1,$$

One can rewrite this general equation $F(x,y)=0$ in matrix form as-

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + h = 0$$

That is-

$$X^T CX + GX = -h$$

Here $C = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is the coefficient matrix for the quadratic terms and $G=[e \ f]$ the coefficient

matrix for the linear terms in x and y. The X is as defined earlier and the superscript T means the transpose of the matrix. Next using the important rotation matrix properties $M^T=M^{-1}$ and $X'^T=(MX')^T=X'^T M^T$, allows us to write things as follows-

$$X'^T [M^{-1}CM] X' + G M X' = -h$$

This represents the function $F(x',y')=0$ expressed in terms of the new coordinates x' and y' . Its evaluation can readily be carried out by computer once a rotation angle θ has been specified and the values of a, b, c, e, and f are given.

Let us consider several specific examples starting with perhaps the simplest case of a straight line $y=\alpha+\beta x$. Rotating it by θ radians is accomplished by the matrix operation-

$$\begin{bmatrix} \beta & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = -\alpha$$

which can be expanded to yield the new line-

$$\{\beta \sin(\theta) + \cos(\theta)\}y' = \alpha + \{-\sin(\theta) + \beta \cos(\theta)\}x'$$

When $\theta=\pi/2$, this result reduces to $\beta y'=\alpha-x'$ from which we have the well known calculus rule that when two straight lines are orthogonal their slopes are negative reciprocals of each other.

Next we look at a standard ellipse for which we have set $a=(1/\alpha^2)=1/16$, $b=c=f=0$, $c=1$ and $h=-1$. This equation has the matrix form-

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \quad \text{or in its abbreviated version} \quad X^T C X = 1$$

If we rotate things by $\theta=\pi/4$, it produces-

$$M^{-1}CM = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/16 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$$

For this particular rotation angle, we obtain the new 2D curve-

$$\begin{bmatrix} x', y' \end{bmatrix} \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 32$$

This happens to be identical with the quadratic equation in x and y given at the beginning of this note. It tells us that it represents an ellipse of semi-major axis 4 and semi-minor axis 1 rotated by 45 degrees. If we change the rotation angle to another value, a different form for $F(x',y')$ will result.

As a third example look at the 2D function-

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2$$

A rotation by $\theta=-\pi/3$ radians, produces the result-

$$M^{-1}CM = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

so that-

$$X'^T [M^{-1}CM] X' = -2x'^2 + 2y'^2 = -2$$

That is, we have a standard hyperbola-

$$x'^2 - y'^2 = 1$$

which has no $x'y'$ product term. We could also rotate things by an extra $\pi/4$ radians to get the hyperbola $x''y'' = 1/4$.

Finally, consider the parabola $y = x^2$ and ask what will its form be after a θ radian rotation. In matrix form this equation reads-

$$X^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X = [0 \ 1] X$$

Applying the above rotation matrix identities, this matrix form converts to-

$$X'^T [M^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M] X' = [0 \ 1] M X'$$

which may be re-written as-

$$x^2 \cos(\theta)^2 - xy \sin(2\theta) + y^2 \sin(\theta)^2 = x \sin(\theta) + y \cos(\theta)$$

and represents the original parabola rotated by θ radians. We have dropped the primes from the new coordinates to simplify the book-keeping. Rotating $y = x^2$ by $\theta = \pi/4$ radians produces the result-

$$x^2 - 2xy + y^2 = \sqrt{2}(x + y)$$

while a $\theta = \pi/2$ rotation yields the parabola $x = y^2$. A plot of the parabola and its two rotated forms are shown in the following figure-

ROTATED PARABOLA

