SEQUENCES AND THEIR CORRESPONDING DIFFERENCE EQUATIONS

It is well known that there are an infinite number sequences Y whose elements can be represented by a unique point function y(n) which are also expressible as a difference equation. We wish in this article to examine such functions and show how they are derived.

Starting with the simple sequence-

$$Y=\{1, 3, 6, 10, 15, 21, ...\}$$

we can write down the following difference forms-

1	3	6	10	15	21	zeroth difference
2	3	4	5	6		first difference
1	1	1	1			second difference
0	0	0				third difference

Since the differences stop with the third we know that the point function yielding the terms y(n) of sequence Y will have the form-

$$y(n)=An^2+Bn+C$$

, where the constants A, B, and C are given by a solution of the matrix equation -

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Solving, one has A=B=1/2 and C=0. Hence the sequence has individual element values of-

$$y(n)=\frac{n(n+1)}{2}$$

These y(n) values represent the triangle numbers. So the seventh triangle number equals y(7)=28. To get the difference equation for the elements, we can write y(1)=1 and-

$y(2)=3=y(1)+2$
$y(3)=6=y(2)+3$
$y(4)=10=y(3)+4$
$y(5)=15=y(4)+5$

Generalizing, we find the difference equation-

$$y(n)=y(n-1) + n \quad \text{subject to} \quad y(1)=1 \qquad .$$

One can also reverse the procedure by starting with a difference equation such as-

$$y(n)=y(n-1)+2n-1 \quad \text{subject to } y(1)=1$$

Writing out the elements we get-

$$y(2)=4 \text{ , } y(3)=9,\ y(4)=16,\ y(5)=25$$

So the corresponding sequence reads-

$$Y=\{1,4,9,16,25,\ldots\}$$

with the element values given by $y(n)=n^2$.

Consider next the sequence-

$$Y=\{1,2,5,12,27,\ldots\}$$

Its elements have the corresponding difference equation form-

$$y(n)=2y(n-1)+n-2 \text{ subject to } y(1)=1$$

Writing out the differences we find-

1   2   5   12   27   58   121
1   3   7   15   31   63
2   4   8   16   32

One notices that here the second difference goes as 2^n. It suggests the point function-

$$y(n)=2^{n-}+An$$

At n=3, this says $5=2^3+A3$, so that A=-1. Hence we have the point function-

$$y(n)=2^n-n$$

which generates all elements of the sequence Y.

Take next the sequence-

$$Y=\{1,4,11,26,57,\ldots\}$$

This sequence equals the elements encountered by looking at the elements along column two in a modified Pascal Triangle as discussed earlier at (http://www2.mae.ufl.edu/~uhk/MORE-PASCAL.pdf). Writing out the first few differences we find-

$$\begin{array}{ccccccccc} 1 & & 4 & & 11 & & 26 & & 57 \\ & 3 & & 7 & & 15 & & 31 & \\ & & 4 & & 8 & & 16 & & \end{array}$$

This suggests a point function-

$$y(n)=2^{n+1}-(n+2)$$

So, for example, y(3)=11, y(7)=247 and y(10)=2036. To get the corresponding difference equation , we write-

$$y(1)=1,\ y(2)=2y(1)+2,\ y(3)=2y(2)+3, \text{ and } y(4)=2y(3)+4$$

On generalizing we get the difference equation-

$$y(n)=2y(n-1)+n \quad \text{subject to } y(1)=1 \quad ,$$

Next, starting with the generalized difference equation-

$$y(n)=y(n-1)+f(n) \text{ subject to } y(1)=1$$

we find-

$$y(2)=1+f(2),\ \ y(3)=1+f(2)+f(3), \text{ and } y(4)=1+f(2)+f(3)+f(4).$$

This produces the solution-

$$y(n)=1+\sum_{k=2}^{n} f(k)$$

If we now take $f(k)=k^2$, one finds the sequence and element values of-

$$Y=\{1, 5, 14, 30, 55, \ldots\}$$

and

$$y(n)=1+\sum_{k=2}^{n} k^2$$

If instead we take $f(k)=(k-1)^2$, we get the sequence-

$$Y=\{1, 2, 6, 15, 31, 56, \ldots\} .$$

Writing out the differences we have-

1 2 6 15 31 56 92
1 4 9 16 25 36
3 5 7 9 11
2 2 2 2
0 0 0

This suggests $y(n)=An^3+Bn^2+Cn$. On solving the corresponding matrix equation for the constants A, B, and C we find the point function for the elements-

$$y(n)=\frac{1}{8}\left[3n^{3}-7n^{2}+10n\right]$$

Thus, for example, y(10)=2400/8= 300.

Starting with anyone of an infinite number of additional difference equations , one can also generate sequences with fractional elements and ones where y(∞) reaches a finite limiting value. A good example for such a case starts with the difference equation-

$$y(n)=1+\frac{1}{1+y(n-1)} \qquad subject \quad to \quad y(1)=1 \ .$$

Its solution produces the fractional number sequence-

Y={1, 3/2, 7/5, 17/12, 41/29, 99/70, … }

Also it yields the interesting result –

$$\frac{\lim}{n\rightarrow\infty}[y(n)]=\sqrt{2}=1.41421356...$$

An additional well known constant y(∞) is produced by the difference equation-

$$y(n+1)=1+\frac{1}{y(n)} \quad subject \quad to \quad y(1)=1$$

Here we have the equivalent sequence-

Y={1, 2, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21, 55/34, 89/55…}

The elements are easy to construct and seem to be heading towards 1.618 as n gets large. Indeed we find-

y(∞)=φ=[1+sqrt(5)]/2=1.61803398…

with φ known as the Golden Ratio. The ancient Greeks were particularly fond of this irrational constant.

Another difference equation , which we first discovered back in 2012 while studying arctan expansions, is-

$$y(n+1) = y(n)+\cos[y(n)]\{\cos[y(n)]-\sin[y(n)]\} \quad \text{subject to } y(0)=1$$

In just eight iterations of this equation we find $y(8)$ to yield a one-hundred digit accurate estimate for $\pi/4$. We obtain-

$$4\cdot y(8)\approx 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706 8$$

in a split second.

As a final sequence consider the point function

$$y(n)=\sum_{k=0}^{n}\frac{1}{k!}$$

Here we have the sequence, starting with n=0, of-

$$Y=\{1, 2, 5/2, 8/3, 65/24, 163/60, \ldots\}$$

One recognizes at once that $y(n)$ approaches $\exp(1)=2.7182818\ldots$
Also looking the individual terms in this sequence, we have the equivalent difference equation-

$$y(n)=y(n-1)+1/n! \text{ subject to } y(0)=1$$

Note that $y(6)$ equals $163/60+1/720=1957/720$.

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