TAYLOR , MACLAURIN, AND LAURENT EXPANSIONS OF FUNCTIONS

Among the more important equalities encountered in elementary calculus are the finite and infinite Taylor and MacLaurin Series. Both have the form-

$$y(x) = y(a) + y'(a)\frac{(x-a)}{1!} + y''(a)\frac{(x-a)^2}{2!} + y'''(a)\frac{(x-a)^3}{3!} + ... = \sum_{n=0}^{m} y^{(n)}(a)\frac{(x-a)^n}{n!}$$

where the exponent (n) indicates the nth derivative of the continuous function y(x) at x=a. The number of terms in the series will equal m+1 if the function y(x) has no derivatives past n=m. Otherwise one has an infinite series. For a≠0 the series is referred to as a Taylor series while a=0 produces a MacLaurin series. The derivation of this expansion is straight forward. One starts with the polynomial expression-

$$y(x) = \sum_{n=0}^{m} A_n (x-a)^n$$

and differentiates it two times to get-

$$y'(x) = \sum_{n=0}^{m} nA_n (x-a)^{n-1} \quad and \quad y''(x) = \sum_{n=0}^{m} n(n-1)A_n (x-a)^{n-2}$$

On setting x=a, we get $y(a)=A_0$, $y'(a)=1\cdot A_1$, and $y''(a)=1\cdot 2\cdot A_2$. Thus, the Taylor series follows.

Let us now consider several different functions and obtain their series about different chosen points 'a'. Consider first the function y(x)=exp(x). Here the nth derivative just equals the function and we have-

$$\exp(x) = \sum_{n=0}^{\infty} \exp(a)\frac{(x-a)^n}{n!}$$

On setting a=0, we get the familiar MacLaurin series-

$$\exp(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

If we set a=1, we have the Taylor series-

$$\exp(x) = \exp(1)\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

The expansion for exp(x) can be used, for example, to evaluate the error function as follows-

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_{t=0}^{x}\exp(-t^2)dt = \frac{2}{\sqrt{\pi}}\int_{t=0}^{x}(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n})dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

This series converges very rapidly for small x so that erf(0.1)=0.1124629160.. is well approximated by the first three terms of the infinite series which yield 0.1124629187 at the same value of x=0.1.

As another examples of infinite series expansions consider the hyperbolic functions y(x)=sinh(x) and cosh(x). On setting a=0, one finds the expansions-

$$\sinh(x) = \sum_{n=0}^{\infty}\frac{(x)^{2n+1}}{(2n+1)!} \quad and \quad \cosh(x) = \sum_{n=0}^{\infty}\frac{(x)^{2n}}{(2n)!}$$

Thus , on setting x=1, we obtain the identities -

$$\sinh(1) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!} = 1.175201... \; and \; \cosh(1) = \sum_{n=0}^{\infty}\frac{1}{(2n)!} = 1.543080...$$

Note that sinh(1)+cosh1)=exp(1)=e=2.718281828459045235360...

Another good example of a simple Maclaurin series corresponds to the function-

$$y(x) = [x - 2\sqrt{1+x} + 2]$$

Expanding the first few terms about a=0 and then generalizing, produces the series-

$$y(x) = \frac{1}{2\cdot 2!}x^2 - \frac{1\cdot 3}{2^2\cdot 3!}x^3 + \frac{1\cdot 3\cdot 5}{2^3\cdot 4!}x^4 - \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 5!}x^5 + ... = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n)!}{2^{2n}n!(n+1)!}x^{n+1}$$

A plot of this sum matches the function y(x) perfectly. We can also use it to evaluate the root of two by setting x=1/332928. This produces the very rapidly converging series-

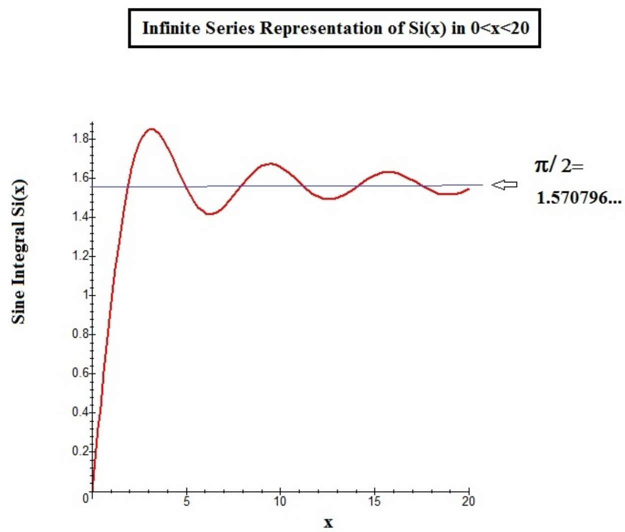
$$\sqrt{2} = \frac{1}{470832}\{665857 + \sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}n!(n+1)!\,332928^n}\}$$

Here just the first term 665857/470832 already approximates sqrt(2) good to eleven decimal places. I leave it to the reader to figure out how I came up with this value for x.

One can also evaluate the sine integral Si(x) using its infinite series expansion. Such an expansion reads-

$$Si(x)=\int_{t=0}^{x}\frac{\sin(t)}{t}dt=\int_{t=0}^{x}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}dt=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

This is made possible by first going into the infinite sum form of sin(t)/t and integrating with respect to t. A plot of this series in the range 0<x<20 follows-



Note that the function approaches a value of π/2 as x→∞.

Another function, whose infinite series form can be used to calculate the value of π, is the arctan function defined as-

$$\arctan(x)=\int_{t=0}^{x}\frac{1}{1+t^2}dt$$

Using the geometric series and integrating term by terms produces the identity-

$$\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$$

On setting x=1, one finds the Gregory formula for π, namely-

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$$

This series unfortunately is extremely slowly convergent and thus is of little use for finding an accurate value of this constant. By drastically reducing the value of x by use of multiple arctan formulas one can get much faster convergence. Here is one form of such an improved formula which we came up with several decades ago-

$$\frac{\pi}{4} = 12\arctan(\frac{1}{38}) + 20\arctan(\frac{1}{57}) + 7\arctan(\frac{1}{239}) + 24\arctan(\frac{1}{268})$$

Many functions have their derivatives above n=m vanish. Such cases will produce finite polynomials representations for y(x). Take the expansion of $y(x)=x^4$ about a=1. It yields-

$$y(x) = x^4 = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + 1(x-1)^4$$

One recognizes at once the coefficients multiplying the $(x-1)^n$ terms as the Binomial Coefficient C(k,n)=k!/[n!(k-n)!]. Hence we have the generalization-

$$x^k = \sum_{n=0}^{k} \frac{k!(x-1)^n}{n!(k-n)!}$$

This is a (k+1) term polynomial in powers of (x-1). Since it is legitimate to replace x by any other variable , we also can have x → x+1 to obtain-

$$(x+1)^k = \sum_{n=0}^{k} C(k,n)x^n$$

which is recognized as just the binomial expansion of the function $y(x)=(1+x)^k$ about x=0.

Another important aspect of series expansions is that they may be used to determine certain polynomials starting with a generating formula. For example, Legendre first expanded the two variable function -

$$y(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}$$

in a power series in t. Working out the first few terms with a=0, one has-

$$y(x,t) = 1 + xt + \frac{(3x^2-1)}{2}t^2 + \frac{(5x^3-3x)}{2}t^3 +$$

The polynomials in x multiplying the various powers of t are recognized to be the familiar Legendre polynomials $P_n(x)$. One can thus write-

$$y(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

This identity also allows one to generate the second order differential equation for Legendre Polynomials. It reads-

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Also one has the recurrence relation-

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - nP_{n-1}(x)$$

Finally, let us look at a series expansion of a function y(x) when it has a singularity of order m at x=b. Such a function can be written as-

$$y(x) = \frac{f(x)}{(x-b)^m}$$

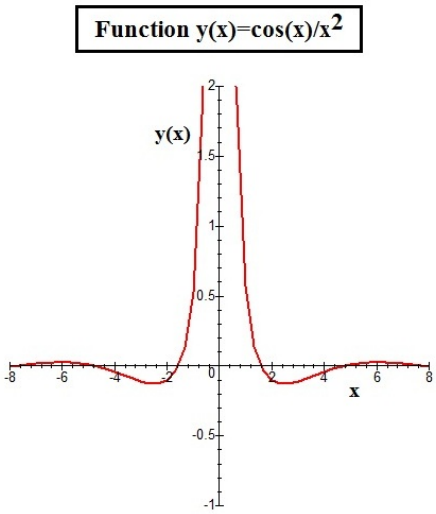
where f(x) is a continuous function and one has an mth order singularity at x=b. On expanding the numerator as a Taylor series about x=b, we get-

$$y(x) = \frac{1}{(x-b)^m}\left\{ f(b) + f'(b)\frac{(x-b)}{1!} + f''(b)\frac{(x-b)^2}{2!} + \ldots \right\}$$

This new series is termed a Laurent Expansion. It blows up at x=b but is a valid representation for all x. A simple example of such a Laurent series is the even function-

$$y(x) = \frac{\cos(x)}{x^2} = \frac{1}{x^2} - \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n x^{2n-2}}{(2n)!}$$

It has a second order pole at x=0 where the function becomes unbounded but yields a finite value elsewhere. A plot of the function looks like this-



The function is seen to vanish at x=(π/2)(2n+1) and becomes unbounded at x=0. One can integrate Laurent expansions of this type to get, for example,-

$$\int \frac{\cos(x)}{x^2}dx = \ln(x) - \frac{1}{2!}x + \frac{1}{3\cdot 4!}x^3 - \frac{1}{5\cdot 6!}x^5 + ...$$

which clearly shows that the area under the curve $\cos(x)/x^2$ is infinite.

The related function y(x)=sin(x)/x , which represents the integrand of the Si(x) integral, is an example of a function with a removable singularity. One has-

$$\int \frac{\sin(x)}{x}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}dx = \frac{x}{1\cdot 1!} - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} -$$

It produces the finite value-

$$\text{Si}(1)= \int_{x=0}^{1} \frac{\sin(x)}{x}dx = 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} ... = 0.9460830704...$$

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