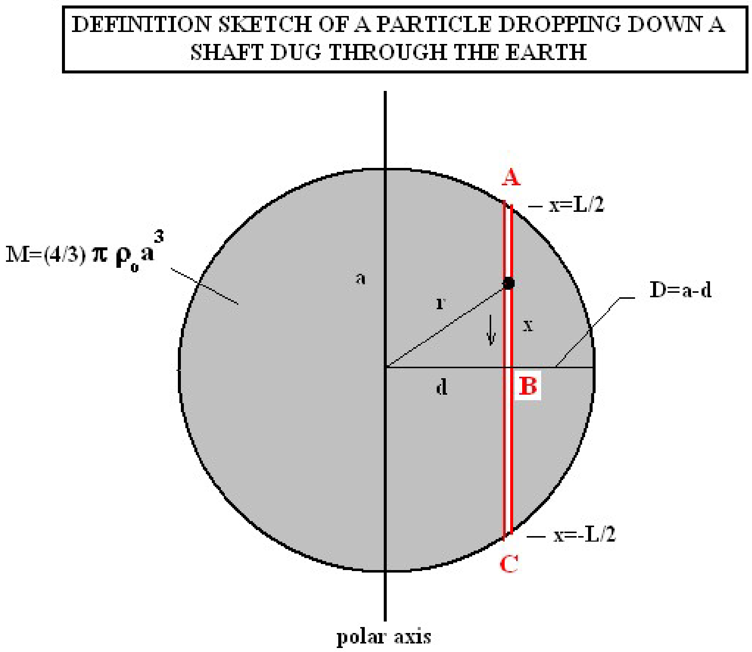
PARTICLE DROPPING THROUGH A SHAFT CONNECTING POINTS A AND C ON THE SURFACE OF THE EARTH

A well known problem encountered by many students in their introductory mechanics class is to determine how long will it take for a particle released from rest at the north pole of the earth to reach the south pole when moving through a straight-line shaft connecting the poles. A variation of this problem which we will consider here is as shown in the following definition sketch-



We are dealing here with a shaft of length L connecting points A and C and a particle of mass m to be released from rest at point A. The simplest way to determine the speed of the mass at point x along the shaft is to treat the problem as one of the conservation of energy and recognize that one is dealing with the interchange of potential and kinetic energies similar to what occurs during the swing of a trapeze artist. In our consideration we make the simplifying assumptions that the earth of radius r=a has constant density ρ_0 and that all friction forces can be neglected. Under those conditions one needs to only consider the kinetic energy $T=(1/2)mv^2$ and the potential energy φ obtained from solving the Poisson equation. The solution for the potential energy inside the earth takes on the well known quadratic form-

$$\phi = -\frac{GMm}{a}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{a}\right)^2\right]$$

,where G is the universal gravitational constant and $M=(4/3)\pi\rho_0 r^3$ the mass of the earth. Note that the potential at the earth's surface becomes $\varphi=-GMm/a=-mga$, where g is the normal acceleration of gravity. The conservation of T plus φ now states that-

$$\frac{GM}{a}=-\frac{1}{2}v^2+\frac{GM}{a}\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r}{a}\right)^2\right]$$

Noting from the definition sketch that $x^2+d^2=r^2$, we find the speed at position x along the shaft to be-

$$v=\frac{dx}{dt}=\sqrt{\frac{GM}{a}}\left[1-\left(\frac{d^2+x^2}{a^2}\right)\right]^{1/2}$$

This result is equivalent to the integral-

$$\int_{x=L/2}^{x}\frac{dx}{\sqrt{(a^2-d^2)-x^2}}=\frac{1}{a}\sqrt{\frac{GM}{a}}\int_{t=0}^{t}dt$$

and on integration it yields-

$$x=\sqrt{a^2-d^2}\cos\left[\frac{1}{a}\sqrt{\frac{GM}{a}}t\right]$$

Thus the mass undergoes a simple harmonic motion at angular frequency $\omega=\text{sqrt}(GM/a^{3/2})=\text{sqrt}(g/a)$. The period of one complete cycle will be-

$$\tau=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{a}{g}}$$

Substituting the values of $g=9.81\text{m/s}^2$ and $a=6.371\text{x}10^6$ m valid for the earth, one finds that the round trip takes $\tau=5063\text{s}=1\text{hr}$ and 24min. What is most interesting about this result is that the time to travel from point A to point C is always the same value of $0.5\tau=42$ minutes <u>regardless of the value of d</u>. The maximum speed is reached at point B and will be-

$$v=\sqrt{g}\left[a^2-d^2\right]^{1/2}$$

The acceleration felt by the mass m will be-

$$\frac{d^2x}{dt^2} = -\omega^2\sqrt{a^2 - d^2}\cos(\omega t)$$

and the mass has a maximum acceleration of -

$$A_{Max} = g\ \mathrm{sqrt}[1-(d/a)^2]$$

at t=0 and t=(1/2)τ. This is a reasonably low value and so can easily be tolerated by humans.

Consider now the potentially practical case where L/a<<1. In this case the maximum depth D=a-d which the shaft lies below the earth surface will be-

$$D = a - d = a\left[1 - \sqrt{1 - (\frac{L}{2a})^2}\right] \approx \frac{L^2}{8a}$$

Thus a 100 mile long shaft drilled between point A and C will have a maximum depth of 1650 ft at B. The speed at the deepest point will be-

$$v_{\max} = \omega\sqrt{a^2 - d^2} = \frac{L}{2}\sqrt{\frac{g}{a}}$$

For the same L=100 mile long shaft this gives the maximum attained speed of 325 ft/sec. If one doubled the length of the shaft then the maximum depth D will increase by a factor of four while the maximum speed doubles.

We point out that that the maximum speed of a mass dropping through a shaft going through the center of the earth will be sqrt(ga) and thus matches the orbital speed of a near earth satellite.