SOLUTION OF THE NON-LINEAR ALGEBRAIC EQUATION $x^a=b^x$.

A problem appearing on many mathematics olympiads throughout the world is to find the analytic solution to-

$$x^a=b^x$$

, where a and b are specified positive integers and only real x solutions are considered. We want to show here how this equation can be quickly solved using the Lambert Function.

The solution starts with taking the natural logarithm of both sides to yield-

$$\ln(x)/x = (1/a)*\ln(b)$$

Next let $u=\ln(x)$ which produces $x=\exp(u)$. Substituting this new variable into the equation produces-

$$-u \exp(-u)=-(1/a) \ln(b)$$

Now recalling that the Lambert Function W has the property that-

$$W(c \exp c)=c$$

, we get-

$$u=\ln(x)=-W((-1/a)\ln(b))$$

Solving for x then produces the closed form solution-

$$x= 1/[\exp(W(-\ln(b)/a)\ )]$$

The internet, especially u-tube, contains many solutions for specified a and b. These include solutions to $x^4=3^x$, $x^2=2^x$, and $x^5=8^x$. Two especially

attractive female presenters on this topic can be found via a google search using the words-

ivana lambertW function or nancypi-inverse functions

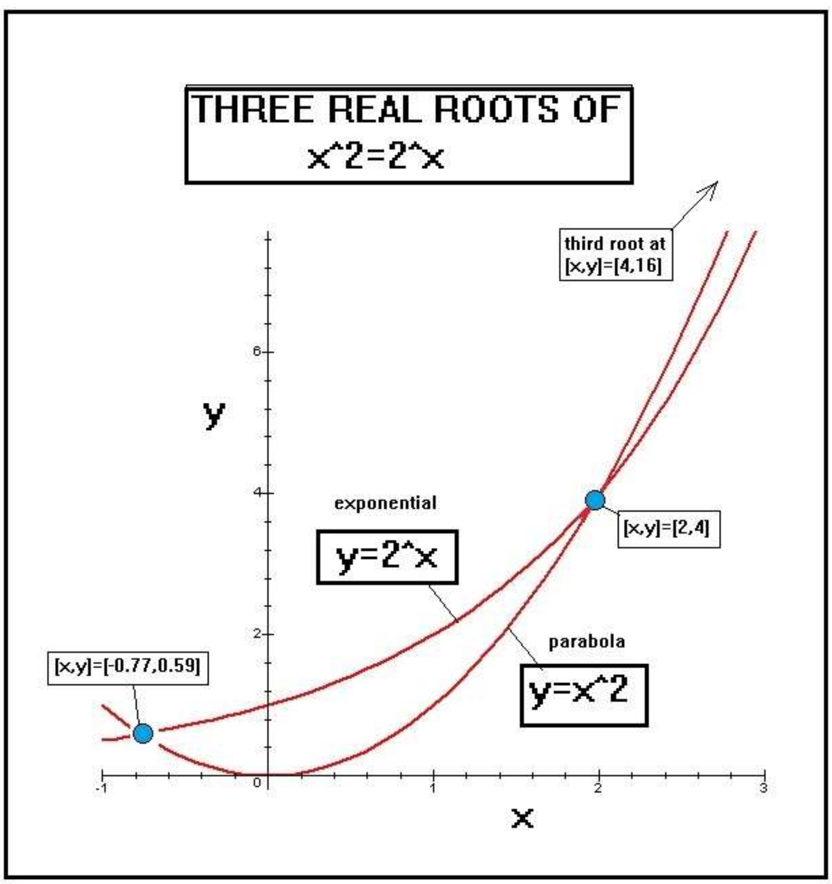
We want to next dissect one of these equations to determine more details. The special case being considered is the non-linear algebraic equation-

$$x^2=2^x$$

, where a=b=2. It has one obvious solution -

$$x=1/\exp(W(-\ln(2)/2))=\exp^(\ln(2)=2$$

Is this the only solution? To answer this point we draw a graph of both sides of this restricted equation. The picture looks like this-



There are a total of three real solutions with the one at [x,y]=[2,4] being given by the above LambertW solution. The oher two can also be recovered by looking at the proper branch of this function. What needs to be remembered is that the LambertW function is the inverse of xexp(x) . This means that one always has the basic identity-

$$W(c\exp c)=c$$

for any c. Note that here the other two solutions fall near [x,y]=

[-0.77,0.59] and at [x,y]=[4,16]. These numbers were obtained by focusing our plot near these points. Such a numerical approach does not use the W function. It often will be the preferred route when one is only interested in a specific final numerical result.

U.H.Kurzweg
July 4, 2024
Gainesville, Florida