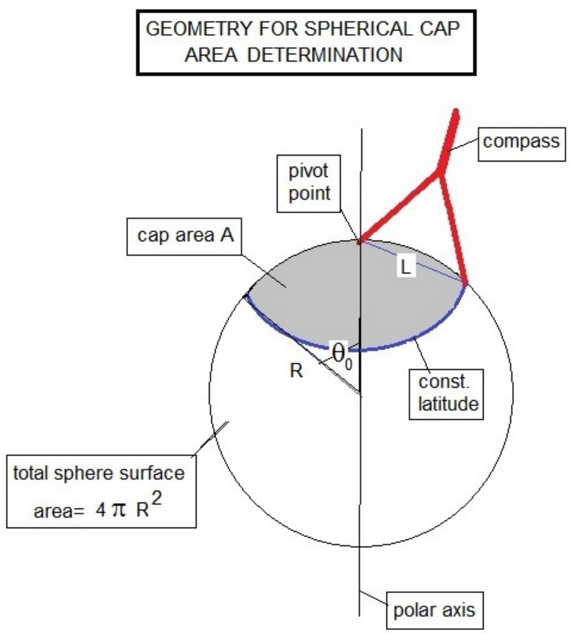
AREA OF A SPHERICAL CAP

In yesterday's issue (Dec.23,2017) of the Wall Street Journal a problem was posed in their math recreation section which asks one to find the area of a spherical cap drawn by a compass opened to a set radius L on the surface of both the earth and the sun. The answer to this problem is straight forward and involves just a little geometry and calculus involving spherical coordinates. Even without any detailed mathematics one knows that if the spherical cap covers half a sphere of radius R, its surface area will be $2\pi R^2$. For this case we also have that the compass opening L will equal sqrt(2)R so that it produces a circle of the same area $2\pi R^2$. This means that no matter what the radius of the sphere being used is, the compass sweeps out the same surface area of πL^2 .

We next want to replace these observations with a more detailed analysis. Our beginning point for such an analysis is the following geometrical sketch-



We have here a sphere of radius R and surface area $4\pi R^2$. A large compass is spun around the north pole of the sphere with a radius setting of L. This maneuver produces a spherical cap on the sphere surface. The lower outer edge of the cap lies at latitude LAT = $(\pi/2-\theta_0)$. The value of θ_0 is determined by the isosceles triangle R-L-R. We have-

$$\sin(\frac{\theta_0}{2}) = \frac{L}{2R} \quad so \quad that \quad \theta_0 = 2\arcsin(\frac{L}{2R})$$

Now to get the spherical cap area we employ spherical coordinates with θ the angle relative to the polar axis and φ the azimuthal coordinate about the polar axis. An increment of area on the spherical cap becomes-

$$dA = R^2 \sin(\theta) d\theta d\varphi$$

On integrating we obtain the spherical cap area-

$$A = R^2 \int_{00}^{\theta_0} \sin(\theta) d\theta \int_0^{2\pi} d\varphi = 2\pi R^2 \{1 - \cos(\theta_0)\}$$

We evaluate this result by expanding arcsin(x) as the series-

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + ...$$

And then evaluating the series form of cosine. This produces-

$$A = 2\pi R^2 [2(\frac{L}{2R})^2] = \pi L^2$$

So as surmised earlier, the cap area is independent of the sphere radius and goes as the square of the compass radius setting of L. This result also allows us to state that-

$$\cos(\theta_0) = 1 - \frac{1}{2}(\frac{L}{R})^2$$

We can now apply the above results to that of the earth and sun configuration considered in the WSJ puzzle page. We have the following hand-book data-

L=1000miles=$1.609x10^6$m=compass radius
r=$6.380x10^6$ m=Earth radius
R=$6.96x10^8$m=Sun radius

These numbers produce the result –

$$A_{sun} = A_{earth} = \pi L^2 = 8.1332x10^{12} m^2 = 3.14159x10^6 sq.miles$$

Note that the spherical cap area on the earth covers only about 1/64 of the entire earth surface .

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