

PROPERTIES OF SPIRALS

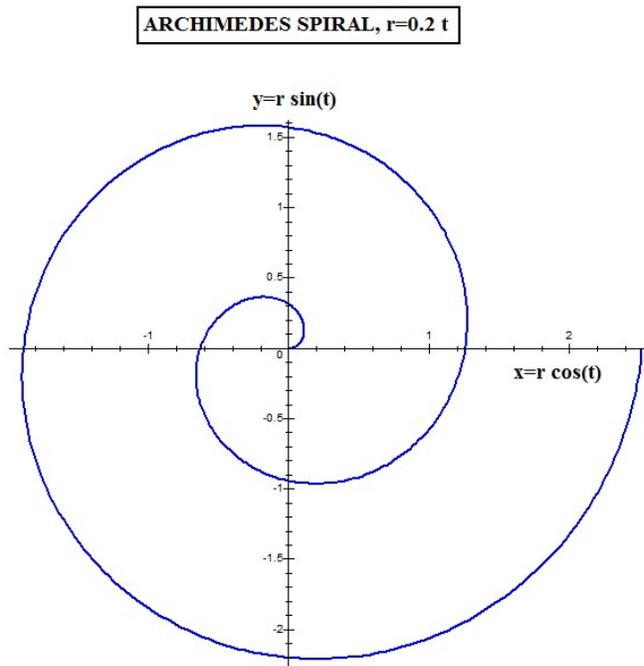
One of the more interesting 2D mathematical curves is the spiral defined by-

$$r = f(\theta) \quad \text{where } r \text{ and } \theta \text{ are polar coordinates}$$

Generally one wants to have $f(\theta+\epsilon) > f(\theta)$ for an outward winding spiral and the reverse for an inward winding spiral. There are an infinite number of such spirals, a fraction of these have names of mathematicians attached to them. The most elementary spiral is that of Archimedes which reads-

$$r = a\theta \quad \text{or} \quad \sqrt{x^2 + y^2} = a \arctan\left(\frac{y}{x}\right) \quad \text{with 'a' const.}$$

Its plot looks like this-



when $a=1/5$. The length along this curve from the origin to polar angle $t=\theta$ is-

$$S = \int_{t=0}^{\theta} \sqrt{dr^2 + (rdt)^2} = a \int_{t=0}^{\theta} \sqrt{1+t^2} dt = \frac{a}{2} \left\{ \theta \sqrt{1+\theta^2} + \ln(\theta + \sqrt{1+\theta^2}) \right\}$$

The radius of curvature at $t=\theta$ is-

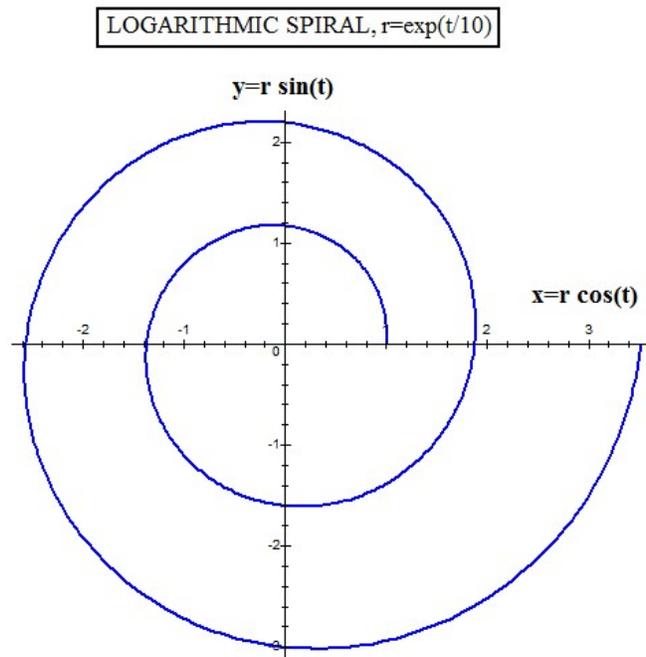
$$R = \frac{[r^2 + (dr/d\theta)^2]^{3/2}}{[r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)]} = a\sqrt{1 + \theta^2}$$

and so increases with increasing angle as the graph indicates.

The next simplest spiral is that of Bernoulli, also known as the logarithmic spiral. It reads-

$$r = \exp(a\theta) \text{ or its equivalent } \ln(r) = a\theta$$

A plot for $a=0.1$ follows-



This time the radius of curvature at $t=\theta$ is-

$$R = \exp(a\theta) \sqrt{1 + a^2}$$

An interesting new feature is that the angle between a radial line and the tangent to the curve has the fixed angle-

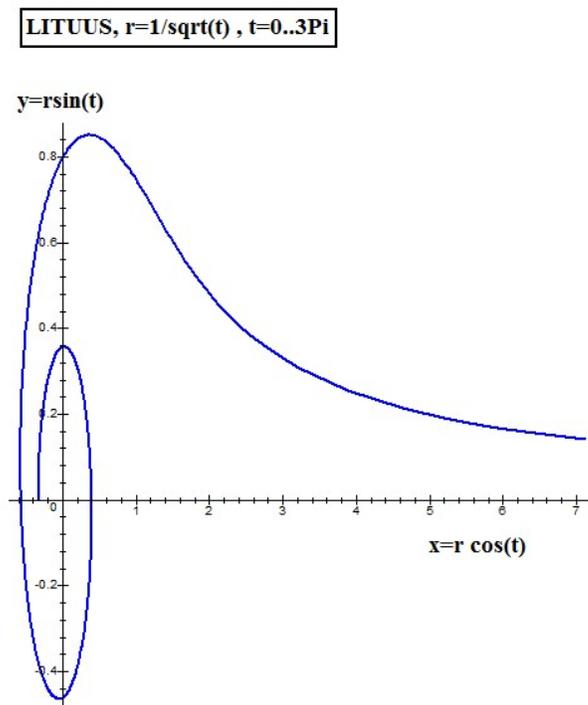
$$\psi = \frac{\pi}{2} - \arctan\left[\frac{dr}{rd\theta}\right] = \frac{\pi}{2} - \arctan(a)$$

I remember visiting J. Bernoulli's grave in Basel, Switzerland several years ago. On his gravestone he had a replica of the logarithmic spiral drawn. Bernoulli believed this spiral to have magical properties.

The next spiral we look at is the Lituus which is really just a subclass of an Archimedes spiral. It reads-

$$r = \frac{1}{\sqrt{\theta}} \quad \text{or} \quad (x^2 + y^2) \arctan\left(\frac{y}{x}\right) = 1$$

and has the graph-



A spiral of this type can be generated by the use of Bessel functions of order $\frac{1}{2}$ and $-\frac{1}{2}$. Letting-

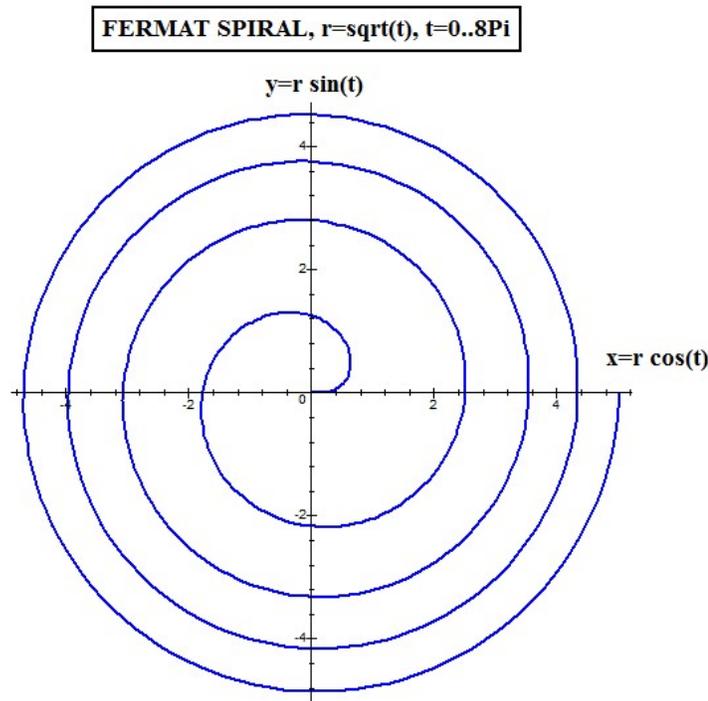
$$x = \sqrt{\frac{\pi}{2}} J_{1/2}(t) = \frac{\sin(t)}{\sqrt{t}} \quad \text{and} \quad y = -\sqrt{\frac{\pi}{2}} J_{-1/2}(t) = \frac{\cos(t)}{\sqrt{t}}$$

with $t=\theta$ being the polar angle and $r=\sqrt{x^2+y^2}$ the radial coordinate.

Another Archimedes like spiral is that of Fermat. It reads-

$$r = \sqrt{\theta} \quad \text{or} \quad \tan(x^2 + y^2) = \frac{y}{x}$$

and looks like this-



This time the distance along the spiral from the origin out to $t = \theta$ is-

$$S = \int_{t=0}^{\theta} \frac{dt}{2\sqrt{t}} \sqrt{1+4t^2} = \int_{z=0}^{\sqrt{\theta}} \sqrt{1+4z^2} dz = \frac{\sqrt{\theta}}{2} \sqrt{1+4\theta} + \frac{1}{4} \ln(2\sqrt{\theta} + \sqrt{1+4\theta})$$

One can generate additional spirals as follows. Begin by specifying the increment of arc length -

$$ds = \sqrt{(dr)^2 + (rd\theta)^2} = dr \sqrt{1 + \left(\frac{f}{f'}\right)^2} \quad \text{where} \quad r = f(\theta) \quad \text{and} \quad f' = \frac{dr}{d\theta}$$

If one now demands that ds/dr be a constant c , then one recovers the logarithmic spiral $\ln(r)=\theta/\sqrt{c^2-1}$. Next consider the case $ds/dr=f$. Here we find the differential expression-

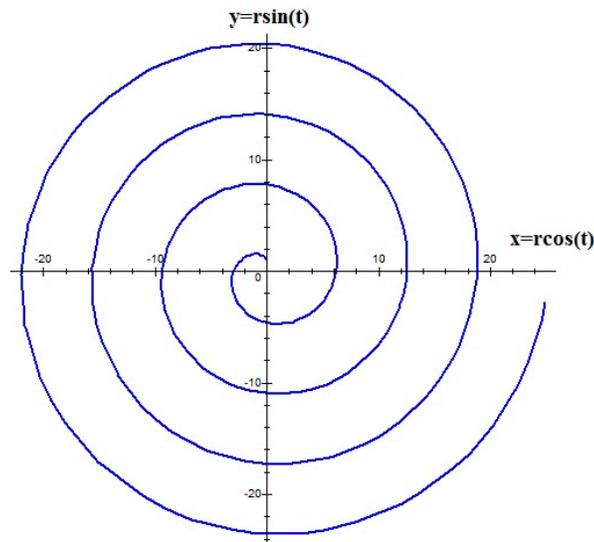
$$\left(\frac{df}{d\theta}\right)^2 = \frac{f^2}{f^2 - 1}$$

which has the implicit solution-

$$\theta = \sqrt{r^2 - 1} + \arctan\left(\frac{1}{\sqrt{r^2 - 1}}\right) \quad \text{with } r > 1$$

and leads to something quite close to an Archimedes type spiral-

THE SPIRAL $t=\sqrt{r^2-1}+\arctan[1/\sqrt{r^2-1}]$



differing only slightly from $r=\theta$.

Many other possibilities exist. For example, one has $r=f=\sinh(t)$ if $ds/dr=\sqrt{1+\tanh(t)^2}$.

Another approach, first used by Euler, is to find the spiral $r=f(\theta)$ for which –

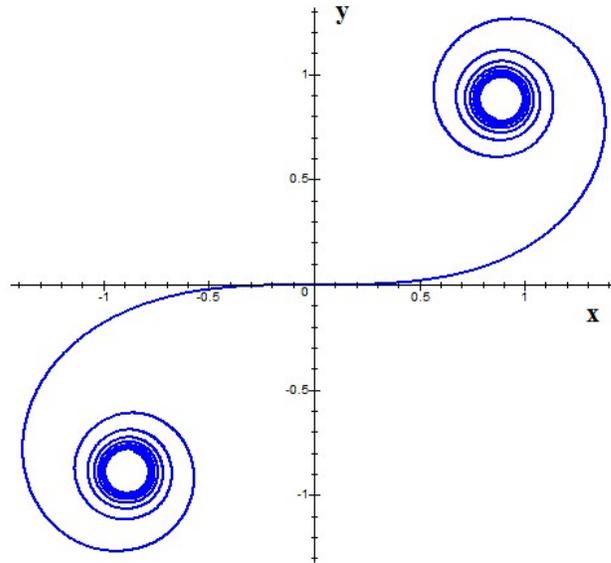
$$\frac{d\theta}{dS} = S \quad \text{or equivalently} \quad \theta = \frac{1}{2} S^2$$

This means-

$$x = \int_{t=0}^s \cos\left(\frac{1}{2}t^2\right) ds \quad \text{and} \quad y = \int_{t=0}^s \sin\left(\frac{1}{2}t^2\right) ds$$

where again $ds = dr \{\sqrt{1+(f/f')^2}\}$. Plotting this curve for $-10 < S < 10$ yields the result-

EULER(OR CORNU) SPIRAL

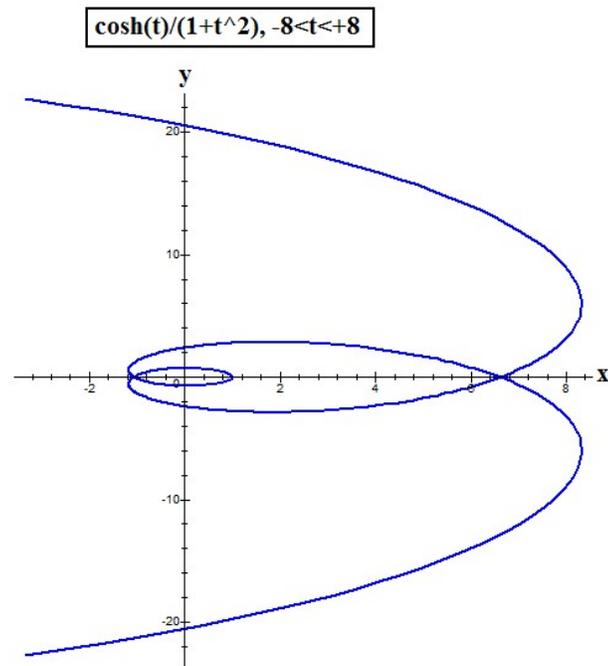


This particular spiral, which is known as either an Euler or Cornu spiral, finds important applications in optical diffraction theory and the above integrals representing x and y are referred to in the literature as Fresnel integrals.

As mentioned above, there are an infinite number of other spirals. Lets look at one more case not found in the literature. It has the form-

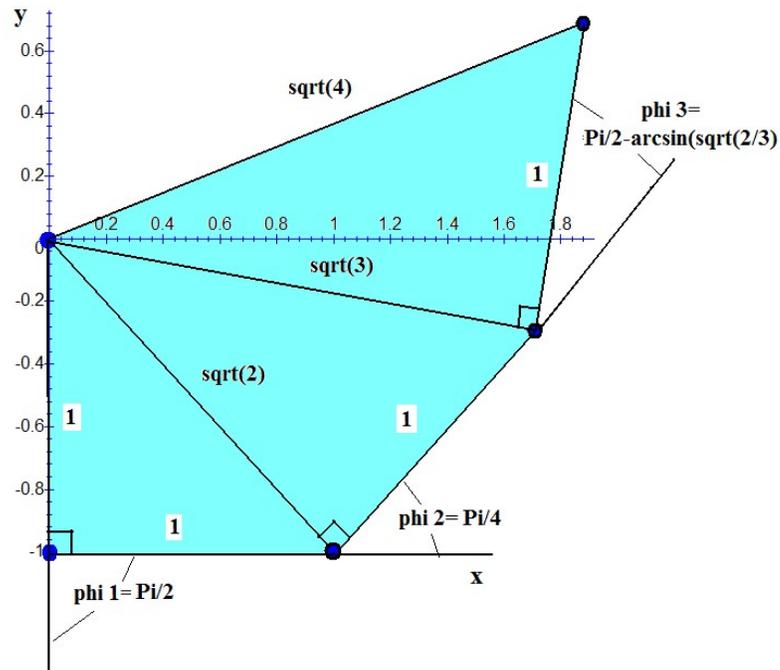
$$r = \exp(-t^2)/(1+t^2) \quad \text{with} \quad -8 < t < 8$$

It has the shape of a double-looped curve as indicated-



We conclude our discussion on spirals by looking at a couple of cases where the derivatives $DS/d\theta$ are not continuous. The first example is the Spiral of Cyrene (also known the Root Spiral). This spiral was first studied by the Greek mathematician Theodorus of Cyrene (5th century BC) and consists of straight line elements of unit length. Each segment is connected at its ends to radial lines of length \sqrt{n} and $\sqrt{n+1}$ forming contiguous right triangles. The triangles are stacked as shown in the following graph-

CONSTRUCTION OF THE SPIRAL OF CYRENE



with the corners indicated as blue dots. From the geometry one notes that the external angle at corner $[x_n, y_n]$ is given as-

$$\varphi_n = \varphi_{n-1} - \arcsin\left(\sqrt{\frac{n-1}{n}}\right)$$

with the coordinates of the nth corner of the spiral are given as-

$$x_n = x_{n-1} + \cos\left(\sum_{k=2}^{n-1} \varphi_k\right) \quad \text{and} \quad y_n = y_{n-1} + \sin\left(\sum_{k=2}^{n-1} \varphi_k\right) \quad \text{when } n \geq 3$$

We have, for example, that-

$$x_4 = 1 + \frac{1}{\sqrt{2}} + \cos\left[\frac{3\pi}{4} - \arcsin\left(\sqrt{\frac{2}{3}}\right)\right] = 1.87620875991..$$

and-

$$y_4 = -1 + \frac{1}{\sqrt{2}} + \sin\left[\frac{3\pi}{4} - \arcsin\left(\sqrt{\frac{2}{3}}\right)\right] = 0.692705340839..$$

As expected the sum of the squares of x_4 and y_4 is equal to 4.

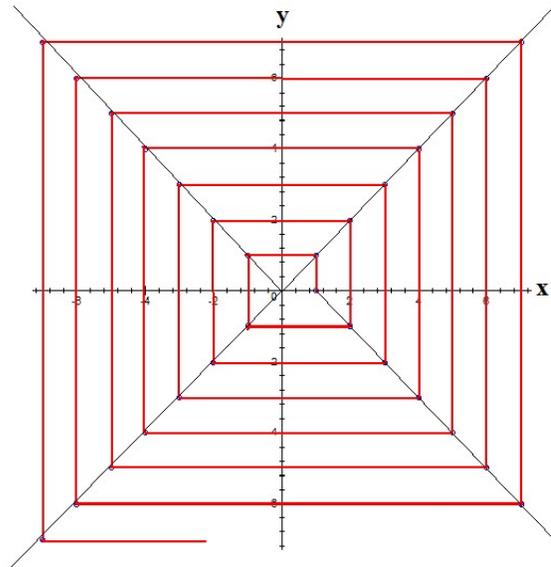
As a second example of a spiral with discontinuities in its derivative, consider the curve generated by connecting points $[1,0],[1,1],[-1,1],[-1,-1],[2,-1],[2,2],[-2,2],[-2,-2],[3,-2],..$ in that order with straight lines. The points lie along four diagonal lines and their coordinates are-

$[(n+1), (n+1)]$ along the diagonal $y=x>0$; $[-(n+1), (n+1)]$ along the diagonal $y=-x>0$;

$[-(n+1), -(n+1)]$ along the diagonal $y=x<0$; $[(n+1), -n]$ along the line $y=(1-x)<0$.

The values of n are taken as 0, 1, 2, 3, etc. Here is the graph-

SIX TURNS OF THE STRAIGHT-LINE SEGMENT SPIRAL



I have indicated the diagonals along which the corners of the spiral are located. Note that it is the displaced diagonal line in the fourth quadrant which makes the generation of the spiral possible. Some observed spiral patterns in crystals which look like the above probably have their origin in the presence of dislocations within the crystal.

