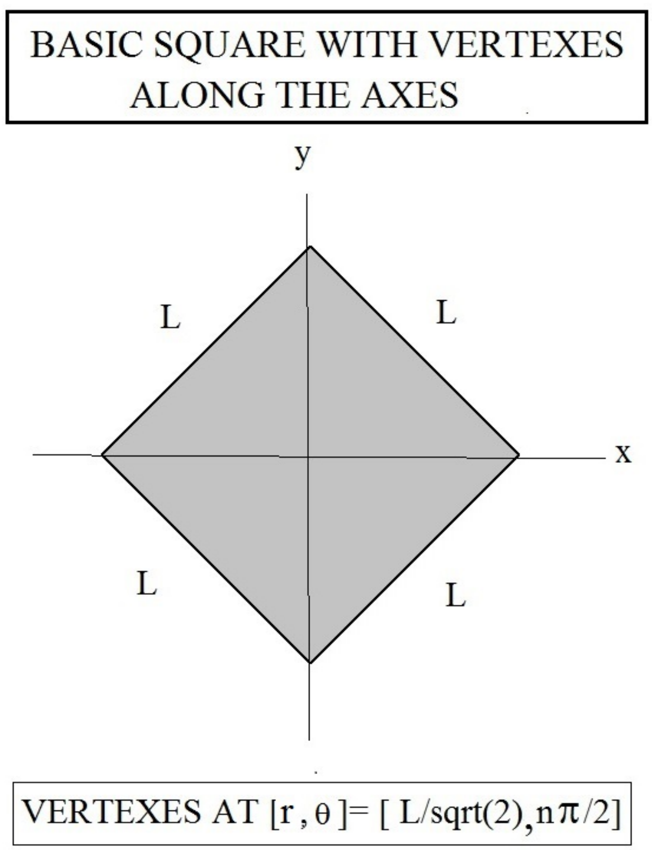
PROPERTIES OF SQUARES

The square is one of the simplest two-dimensional geometric figures. It is recognized by most pre-kindergartners through programs such as Sesame Street, the comic strip Sponge Bob Square Pants, or simply playing with blocks. We want here to look at the general properties of the square and develop some of the relevant formulas describing areas involving this figure.

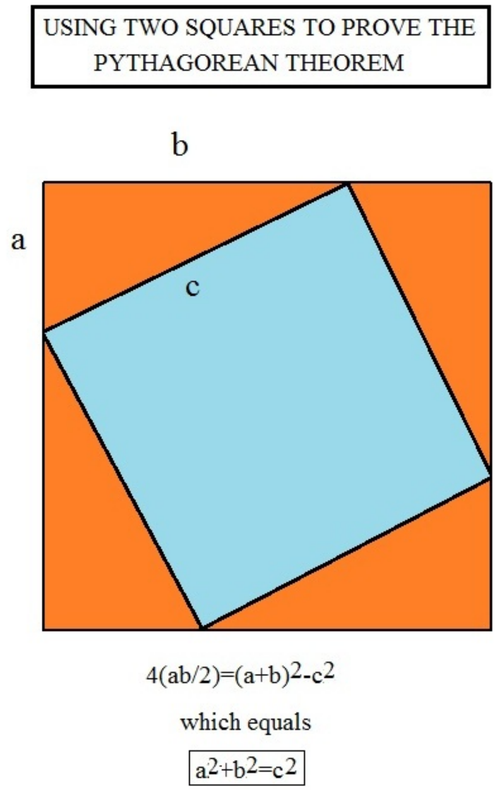
The simplest definition of a square is that it is a figure consisting of four straight line edges of length L each connected to each other by four right angle vertexes. Expressed in terms of side-length L and connecting angle θ, the four vertexes are found at-

$$[L,\theta]=[L,\ \theta_0+(n-1)\pi/2] \quad ,\ n=1,2,3,4$$

Here θ_0 represents the angle the n=1 side makes with respect to the x-axis. Setting $\theta_0 = -\pi/4$ we get the following figure-



The area of this square is simply $A_S=L^2$ with the distance to any of the vertexes from the square center being L/sqrt(2). If we place a smaller square of sides c inside a larger square of sides L=a+b such that the vertexes of the smaller triangle just touch the sides of the larger triangle, we get the following pattern-



From the figure we see that the area of the four orange right triangles just equals the difference in area of the larger to the smaller square. Hence we have-

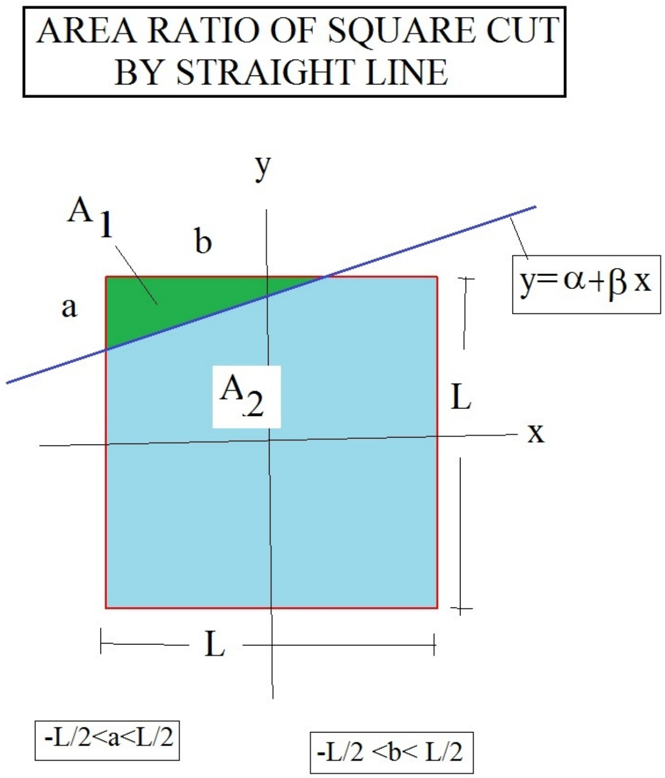
$$4\left(\frac{ab}{2}\right)=(a+b)^2-c^2$$

On multiplying things out we arrive at the famous Pythagorean Theorem –

$$a^2+b^2=c^2$$

Pythagoras of Samos (380-300BC) was a famous Greek mathematician who founded a school which attempted to connect music, the movement of stars and planets, and the fate of man to numbers. It is not sure whether Pythagoras arrived at his theorem from earlier Babylonian work or whether it was an independent discovery. Nevertheless, his announcement called for a major celebration at his school including the sacrificing of an oxen.

One can use the Pythagorean Theorem to determine the ratio of the two parts of a square created by cutting the square by a straight line $y=\alpha+\beta x$ which intersects two neighboring edges. One has the following picture-



Here the area $A_1=ab/2$ and the area $A_2=L^2-A_1$. Hence the ratio-

$$R=\frac{A_1}{A_2}=\frac{1}{[\frac{2L^2}{ab}-1]}$$

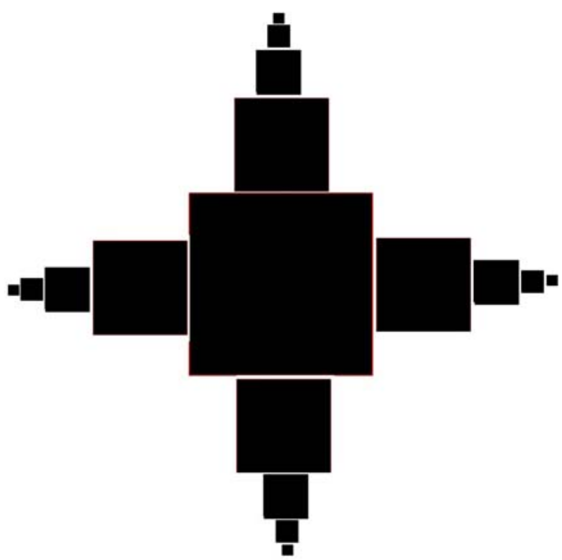
This ratio becomes R=1 when a=b=L as expected. Under that condition the two areas are equal so that the resultant diagonal has length Lsqrt(2) according to the Pythagorean theorem. Should the straight line cross opposite sides of the square the sub-areas will form two trapezoids. The area ratio becomes-

$$R = \frac{2L}{(c+d)} - 1$$

Here c is the distance on the left side of the square to that at the right side, both measure from the line y=-L/2 up. When c=d=L/2, one gets two equal area rectangles of area $L^2/2$ each. The dividing line will have length L.

We next look at some patterns constructed form a collection of many squares. One of the first which comes to mind is to start with a unit side-length square and superimpose a first generation of four smaller squares of side-length ½ as shown. Follow this with a second, third, and fourth generation each time halving the square side-length. This produces the pattern-

FOUR GENERATIONS OF HALVING SQUARE SIDE-LENGTH



This pattern exhibits a four-fold symmetry and is of finite size. The total area of the pattern will be-

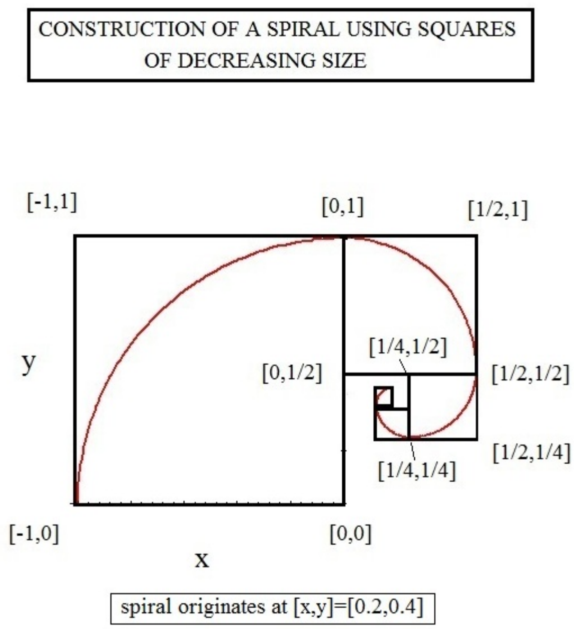
$$\text{Area=} = 1 + 4\{\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}\} = \frac{149}{64} = 2.328125...$$

when taken through the fourth generation. One can extend the calculation to the infinite generation in which case we recognize the area has the slightly larger value of-

$$\text{Area total}==1+\sum_{n=0}^{\infty}\frac{1}{4^n}=\frac{7}{3}=2.333333...$$

The entire pattern just fits into a large square of side-length 3 and total area 9. From this we also see that the black squares take up an area of 7/3 compared to the remaining area inside the large outer bounding square of 20/3.

Another interesting pattern is found by adding onto a unit side-length square a smaller square of side-length ½ at its upper right. Follow this by adding a third square of side length ¼ as shown. Continuing the process each time halving the side-length for the next square, one produces the following pattern-



It is now possible to place quarter circles of decreasing radius into these squares to get the spiral structure shown. This spiral is continuous and has the same derivative in going between squares but has a change in curvature at these transitions This same shortcoming is encountered with the Fibonacci Spiral which looks very much as the present spiral but starts by breaking a rectangle into squares of side-length matching the golden ratio φ=[1+sqrt(5)]/2. In examining the above picture, one could ask to what point does the spiral converge. Finding the precise value first stumped us. However, an hour of thought made us realize that the solution is straight forward. To get the x value you simply start adding up all the x movements starting with x=0. This yields-

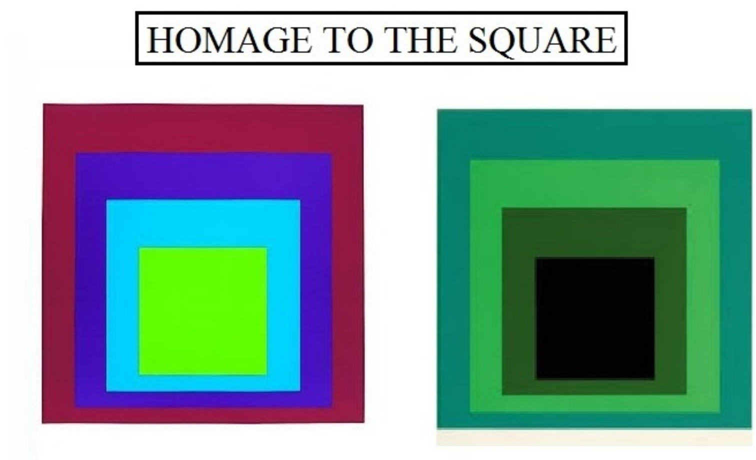
$$x = 1/4 - 1/16 + 1/64 - 1/256 + = (1/4)\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n} = \frac{1}{5}$$

The y value is gotten by the following addition-

$$y = 1/2 - (1/8)\{1 - 1/4 + 1/16 - ..\} = \frac{1}{2} - \frac{1}{8}\sum_{n-0}^{\infty}\frac{(-1)^n}{4^n} = \frac{2}{5}$$

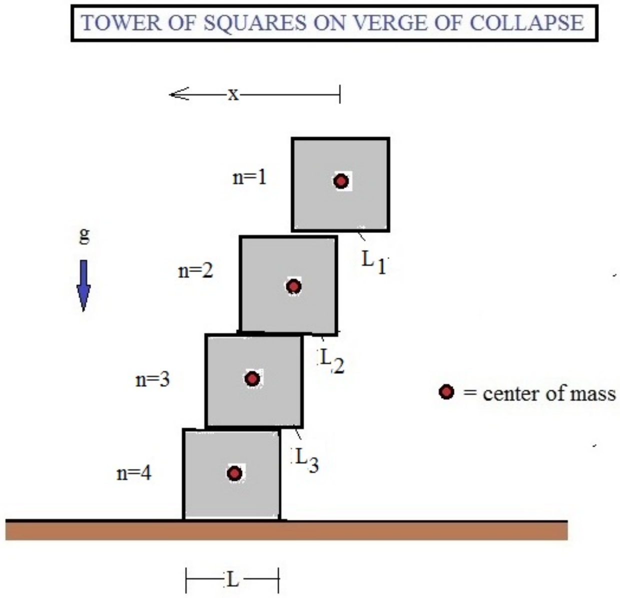
Note that the convergence point [x,y]=[0.2,0.4] of the spiral lies along a radial line passing through the origin [0,0] and inclined at angle arctan(2) relative to the x axis.

The square also makes its appearance in architecture and art. The modern artist who was especially concerned with squares was Joseph Albers. Here are two examples of his work-



Joseph Albers (1888-1976)

We conclude our discussions on squares by looking at an old problem students encounter in their elementary statics course. The question is what is the minimum overlap required between stacked square blocks to keep a tower of such blocks from toppling. Any child playing with blocks will have figured out the solution intuitively but it is not until students become familiar with centers of mass , moments, and reaction forces that they will be able to give a mathematical proof for what it takes to keep the tower stable. We will assume that we are dealing with a set of cubical wooden blocks of equal size and uniform density, Also we will consider only stacking in a fixed vertical plane. Clearly for two blocks of edge length L the impending toppling occurs when the center of mass of the upper block lies just above the edge of the lower block. That is, a L/2 overhang is the maximum allowed. On adding more square blocks, the requirement is that the effective center of mass of the upper n blocks lie exactly above the edge of the n+1 square. We have a picture which looks like this-



The origin of the x coordinate coincides with the center of mass of the first block so that the mass center of the second block lies at x_2=L-L_1= L/2. At impending toppling L_1=L/2. Now to determine the effective mass center in the x direction of the first n blocks we use the formula-

$$\bar{x}_n = \frac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k} \qquad with \quad all \quad m_k = m$$

So for the first two blocks we have –

$$\bar{x}_2 = \frac{[0+\frac{L}{2}]}{2} = \frac{L}{4} = L_2$$

and for the first three blocks we find-

$$\bar{x}_3 = \frac{[0+\frac{L}{2}+\frac{3L}{4}]}{3} = \frac{5}{12}L = L_3 + L_2 + L_1 - L/2 \quad so \quad L_3 = L/6$$

From the trend, we expect L_4 to go as L/8. To test this out we write-

$$\bar{x}_4 = \frac{[0 + \frac{L}{2} + \frac{3L}{4} + \frac{11L}{12}]}{4} = \frac{13}{24} L = L_4 + L_3 + L_2 \quad so \quad L_4 = L/8$$

We can generalize these results by noting the maximum overhang of the group of n top blocks at the nth block relative to the n+1 block will be L/2n. This result shows that the present tower can have a total overhang greater than L since the harmonic series diverges. Indeed any tower of five or more blocks will a shift in x between the top and bottom block of more than L.

January 20, 2017
Inauguration Day