THE MATHEMATICS BEHIND SUDOKU

Sudoku is one of the more interesting and potentially addictive number puzzles. It's modern version (adapted from the Latin Square of Leonard Euler) was invented by the American Architect Howard Ganz in 1979 and brought to worldwide attention through promotion efforts in Japan. Today the puzzle appears weekly in newspapers throughout the world. The puzzle basis is a square n x n array of elements for which only a few of the n^2 elements are specified beforehand. The idea behind the puzzle is to find the value of the remaining elements following certain rules. The rules are-

(1)-Each row and column of the n x n square matrix must contain each of n numbers only once so that the sum in each row or column of n numbers always equals-

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(2)-The n x n matrix is broken up into $(n/b)^2$ square sub-matrixes, where b is an integer divisor of n. Each sub-matrix must also contain all n elements just once and the sum of the elements in each sub-matrix must also equal n(n+1)/2.

It is our purpose here to look at the mathematical aspects behind Sudoku solutions. To make things as simple as possible, we will start with a 4 x 4 Sudoku puzzle of the form-

$$\begin{bmatrix} 1 & - & 3 & - \\ - & - & 1 & - \\ 4 & - & - & 3 \\ - & 2 & - & - \end{bmatrix}$$

Here we have 6 elements given at the outset and we are asked to find the remaining 10 elements shown as dashes. There are four rows and four columns each plus 4 sub-matrixes of 2 x 2 size which read-

$$A = \begin{bmatrix} 1 & - \\ - & - \end{bmatrix} \quad B = \begin{bmatrix} 3 & - \\ 1 & - \end{bmatrix} \quad C = \begin{bmatrix} 4 & - \\ - & 2 \end{bmatrix} \quad and \quad D = \begin{bmatrix} - & 3 \\ - & - \end{bmatrix}$$

Any element in the 4 x 4 square matrix is identified by the symbol $a_{i,j}$,where i is the row number and j the column number. Thus $a_{4,2}=2$ and $a_{2,3}=1$. In solving, one usually starts with unknown elements which are present in the 4x4 array but not present in a chosen sub-matrix. For sub-matrix C there are no elements 1 or 3 present. Looking at the rows and columns passing through C plus sub-matrix C itself, it is seen that $a_{3,2}=1$ and $a_{4,1}=3$ by Sudoku rules. Hence all elements of C have now been determined. Finding the remaining unknowns now quickly follow to produce the completed puzzle-

$$\begin{vmatrix} 1 & 4 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{vmatrix}$$

We note the sum in each row, column and sub-matrix equals 4(4+1)/2=10 as expected. Also each row, column, and sub-matrix contains each of the four elements just once.

Instead of numbers one can also use letters in Sudoku puzzles. For instance, one has the solution-

FOUR BY FOUR SUDOKU LETTER PUZZLE

A	B	D	C
C	D	B	A
D	C	A	B
B	A	C	D

black-given values **red-calculated values**

The above 4 x4 matrices are rather trivial examples of standard Sudoku puzzles which typically involve square matrixes of 9 x 9=81 elements or even 16 x16 =256 elements.

The most common Sudoku puzzle is an 81 element version where each row, column, and nine element sub-matrixes contains all nine single digit numbers 1 through 9 just once. The number of initially given integer elements are kept low (typically 28 to 34) yet still are large enough to make a complete and unique solution possible. The same rules used above for the 4 x4 Sudoku puzzles continue to apply. Mathematically one writes down all the possible integer values an element $a_{i,j}$ could have by recording all numbers it could not have and then taking the compliment. At the beginning of the puzzle solution the possible values for a given unknown $a_{i,j}$ will typically be multiple, but later in the solution process the number will reduce to just one unique value. It is usually a good idea in a Sudoku puzzle solution to mark only those a_{ij} spaces allowing no more than three possibilities. This avoids clutter. Typically a Sudoku puzzle has its solution develop slowly at the beginning but then accelerates as one nears completion. It is usually a good idea to pick the first $a_{i,js}$ at a location where the largest number of elements in a given sub-matrix are already known.

Let us demonstrate how such a solution proceeds for the following 9x9 element Sudoku puzzle-

COMPLETED SUDOKU PUZZLE

4	1	7	5	2	3	6	9	8
2	5	3	9	8	6	1	4	7
9	8	6	1	7	4	3	2	5
6	9	1	8	5	7	4	3	2
5	3	2	4	6	9	8	7	1
7	4	8	2	3	1	5	6	9
3	7	9	6	1	5	2	8	4
8	6	5	7	4	2	9	1	3
1	2	4	3	9	8	7	5	6

black=initial numbers red=calculated numbers

Here there are 31 initially specified numbers shown in black and 81-31=50 unknown numbers shown in red. There are a total of nine 3 x 3 sub-matrixes. The mathematics for working out these red numbers goes as follows-

(1)-Pick an unknown element whose sub-matrix already contains a large number of knowns. Mark these numbers down, recording each number only once.

(2)-Write down the compliment to this list to see which values for $a_{i,j}$ are possible.

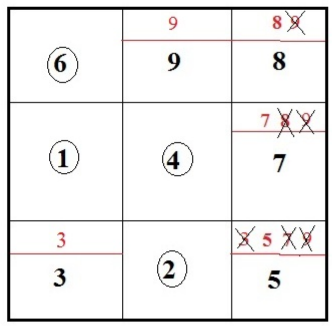
(3) If more than one possible solution arises, keep this information for later evaluation by marking the choices in small letters at the top of the appropriate square. Try to keep the number of choices to a minimum. If there is just one solution mark it down at once to become part of the known element group.

Typically at the beginning of the solution $a_{i,j}$ will have multiple values but as the solution progresses the value becomes a unique number with a value in the range 1 through 9. The sum check of 9(10)/2=45 for elements in any row, column, or sub-matrix provides an accuracy measure. If the same two numbers appear as elements in any row, column, or sub-matrix it is clear that they cannot appear again in any of the remaining spaces in such a row, column, or sub-matrix.

Let us begin our evaluation of the 50 unknowns in the above puzzle. As our first trial consider the elements $a_{3,7}$, and $a_{1,8}$ in the top right sub-matrix. These have the unique single digit values-

$a_{3,7}\rightarrow$comp(1,2,4,5,6,7,8,9)=(3) , and $a_{1,8}\rightarrow$comp(1,2,3,4,5,6,7,8)=(9), Mark these two values in the appropriate places on the puzzle. Next we look at the remaining four unknowns in the sub-matrix .The elimination to get these remaining values looks like this-

EVALUATING ELEMENTS IN UPPER RIGHT SUB-MATRIX



red indicates initial possibilities for complementary numbers

black represents final elements after elimination

circled elements are the initially given values

The unique answers obtained are-

$a_{1,9}=8$, $a_{2,9}=7$, $a_{3,7}=3$ and $a_{3,9}=5$

Going on to the central sub-matrix we find-

$a_{4,5}=(5)$ and $a_{6,5}=(3)$ with $a_{5,4}= a_{5,6}=(4,9)$ remaining unresolved

Record all these unknown into the appropriate places, with the two unresolved values also put into their appropriate positions but with a smaller size font for later partial elimination.

Continuing with the left central sub-matrix we obtain, in the order found,-

$a_{5,3}=(2)$, $a_{5,2}=(3)$, $a_{4,2}=(9)$, $a_{4,1}=(6)$ $a_{6,2}=(4)$, and $a_{6,1}=7$

This completes the evaluation of all unknowns in this sub-matrix. There are many different solution paths one can next take to find the remaining unknowns. We choose to go to the upper left sub-matrix. Evaluating the elements following Sudoku rules produces-

$a_{1,1}=(4)$, $a_{1,2}=(1)$, $a_{2,1}=(2)$, $a_{3,1}=(9)$, and $a_{3,3}=(6)$

Applying these type of procedures to the remaining sub-matrixes then leads to a complete solution shown by the red numbers in the above figure.

It typically takes about 30 minutes to one hour to work out a 81 element Sudoku puzzle of medium difficulty when some 30 elements are given at the beginning. The secret to a quick solution is to spot those spaces which allow only one solution early and then add them to the known elements.

Sudoku rules are simple enough for someone to construct a computer program which will be able to solve any Sudoku puzzle almost instantaneously. I would think such a program could handle puzzles with 16 x 16=256 elements and 25 x 25=625 with ease. A human mind is unlikely to be able to find a complete solution to such a large Sudoku puzzle because of time and patience factors . Computer programs can also be used in reverse to construct Sudoku puzzles. Psychologists have suggested that Sudoku is an excellent numbers game for people of all ages (and especially senior citizens) for keeping the mind sharp and alert.