SUMS AND PRODUCTS OF PRIME NUMBERS

Prime numbers consist of all integers which are divisible by no other numbers than 1 or themselves. Thus 2, 3, 5, 7, 11, are primes. There are an infinite number of these primes all of which are odd numbers except the first. The simple MAPLE command-

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seq{i,(ithprime(i)}, i=1..100);
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produces the sequence pairs-

{1, 2}, {2, 3}, {3, 5}, {4, 7}, {5, 11}, {6, 13}, {7, 17}, {8, 19},

{9, 23}, {10, 29}, {11, 31}, {12, 37}, {13, 41}, {14, 43},

{15, 47}, {16, 53}, {17, 59}, {18, 61}, {19, 67}, {20, 71},

{21, 73}, {22, 79}, {23, 83}, {24, 89}, {25, 97}, {26, 101},

{27, 103}, {28, 107}, {29, 109}, {30, 113}, {31, 127}, {32, 131},

{33, 137}, {34, 139}, {35, 149}, {36, 151}, {37, 157}, {38, 163},

{39, 167}, {40, 173}, {41, 179}, {42, 181}, {43, 191}, {44, 193},

{45, 197}, {46, 199}, {47, 211}, {48, 223}, {49, 227}, {50, 229},

{51, 233}, {52, 239}, {53, 241}, {54, 251}, {55, 257}, {56, 263},

{57, 269}, {58, 271}, {59, 277}, {60, 281}, {61, 283}, {62, 293},

{63, 307}, {64, 311}, {65, 313}, {66, 317}, {67, 331}, {68, 337},

{69, 347}, {70, 349}, {71, 353}, {72, 359}, {73, 367}, {74, 373},

{75, 379}, {76, 383}, {77, 389}, {78, 397}, {79, 401}, {80, 409},

{81, 419}, {82, 421}, {83, 431}, {84, 433}, {85, 439}, {86, 443},

{87, 449}, {88, 457}, {89, 461}, {90, 463}, {91, 467}, {92, 479},

{93, 487}, {94, 491}, {95, 499}, {96, 503}, {97, 509}, {98, 521},

{99, 523}, {100, 541}

for the first hundred primes. Thus the 50s prime is 229 and the hundredth prime is 541.

The spacing between primes becomes larger on average as the value of i increases. The spacing between the 11s and 10^{th} prime is 31-29=2 while the spacing between the 1001 and 1000 prime is 7927-7919=8. Gauss discovered quite early that the number of primes in the integer range $1<n<N$ is-

$$R \approx N/\ln(N)$$

This result is commonly referred to as the Prime Number Theorem and is found to become ever more accurate as N increases. The predicted number of primes in $1<n<500$ is R=500/ln(500)=80.45 while the above sequence shows it to actually be 95. This type of discrepancy will be reduced if one lets N become still larger. Thus the 3000^{th} prime equals 27449 and the corresponding R=2685.79 which lies within $2/10^{th}$ of one percent of the correct answer. The average expected spacing between primes for large N, according to the Prime Number theorem, will be-

$$S = \frac{(N+1)}{\ln(N+1)} - \frac{N}{\ln(N)}$$

This fact however does not exclude the possibility of double primes such as 17957 and 17959.

Another interesting property of primes is that any number which is not a prime can always be reduced to the product of primes. Thus-

$$3487624 = 2^3 \cdot 7^3 \cdot 31^1 \cdot 41^1$$

This type of decomposition which works for every integer allows one to talk about an exponent vector. Thus-

$$\nu(3487624) = [3\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1]$$

where reference to the above sequence pairs has been made. Such exponential vector representations come in handy when decomposing large composite numbers.

Let us next look at some sums and products of the primes. Perhaps the best known historical result is the famous Euler formula-

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \left[\frac{1}{1 - \frac{1}{p_n^{\,s}}} \right]$$

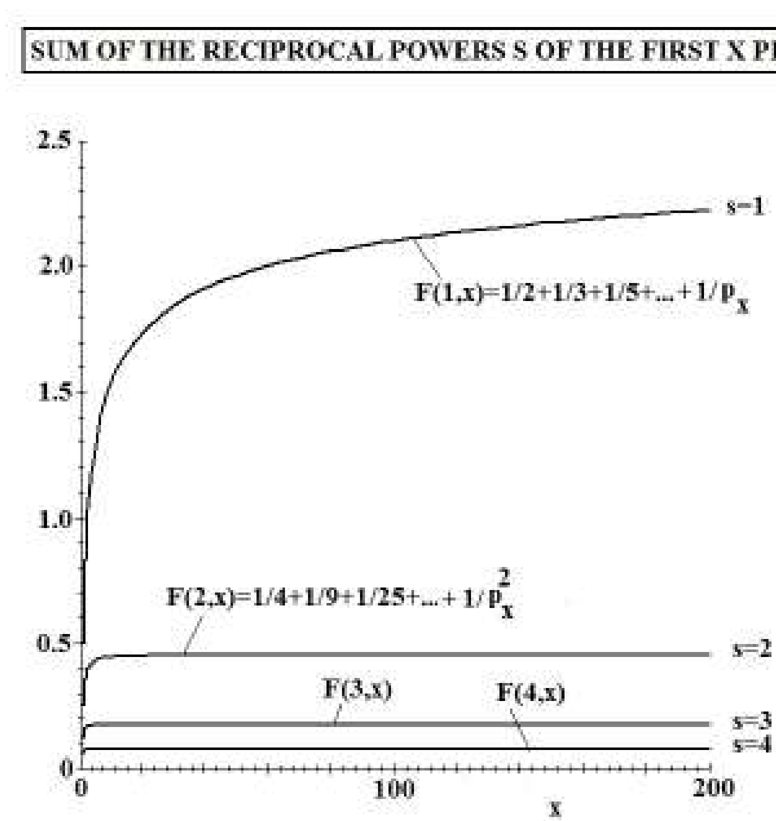
which relates the Zeta function to an infinite product involving the sth power of all primes p_n. Euler derived this remarkable result by the same approach one uses to sum the standard geometric series. The infinite product becomes unbounded when s=1 since it just equals the standard harmonic series under those conditions. However, when s=2, one finds the finite value-

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=(\frac{1}{1-\frac{1}{2^2}})(\frac{1}{1-\frac{1}{3^2}})(\frac{1}{1-\frac{1}{5^2}})(\frac{1}{1-\frac{1}{7^2}})(\frac{1}{1-\frac{1}{11^2}})(....$$

The next identity involving primes which comes to mind is to sum of the reciprocals . We have-

$$F(s,x)=\sum_{n=1}^{x}\frac{1}{p_n^{\ s}}=\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+....+\frac{1}{p^s{}_x}+$$

It is a simple matter to sum this series. The results for s=1, 2, 3, and 4 are shown over the range 1<x<200 in the following graph-



As on lets x approach infinity the series F(s,∞) clearly converge for s=2, 3, and 4 but apparently does not for s=1. Taking things out to x=3000, we find-

$$F(2,x)=0.45224..,\, F(3,x)=0.17476..,\, and\ F(4,x)=0.076993$$

These values lie very close to what the graph shows at x=200 and thus allows us to conclude that they converge to very near these numbers as x approaches infinity. In view of the above Euler result, one would think these numbers are given as some power of π which so far has eluded me. The case of s=1 is a different matter. It appears to be a divergent series diverging even slower than the standard harmonic series reaching a value of just 2.45741.. at x=1000 and 2.58636.. at x=3000. One knows that the sum of the reciprocal of all odd integers diverges and so it is reasonable to also conclude that the sum of the reciprocal of the first powers of all primes also diverges.

Lets next look at the product of the first x primes. We have the function-

$$G(x)=\prod_{n=1}^{x} p_n = (2)(3)(5)(7)(11)(13)(....)(p_x)$$

This function looks like the factorial function after dividing out the non-prime terms 4, 6, 8, 9, 10, 12 etc . Here is a table of the first ten values-

x	G(x)
1	2
2	6
3	30
4	210
5	2310
6	30030
7	510510
8	9699690
9	223092870
10	6469693230

The values increase rather rapidly. One notices after G(2) that all the numbers end in zero. This is expected since once G(3) =2x3x5=30 is reached all subsequent products will need to be multiplied by 3 and 10. So it is not surprising that-

G(100)=471193079990618495316248783476026042202057477340967552018863483961641533584503422120528925670554468197243910409777715799180438028421831503871944494399049257903072063599053845231252833986435299931039848179173001720103109 0 also ends in zero. We can call these G(x) numbers super-composites since they contain every prime number up to and including p_x Their exponent vector consists of all ones and reads-

$$\nu[G(x)]=[11111111...]$$

To express the ks power of G(x) one needs to simply construct the number corresponding to the exponent vector-

$$\nu[G(x)^k]=[k\ k\ k\ k\ k\ k\ k...]$$

This fact allows us to write-

$$G(4)^3 = 210^3 = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 = 8 \cdot 27 \cdot 125 \cdot 343 = 9261000$$

By making use of the properties of logarithms, one also finds that-

$$\ln[G(x)] = \sum_{n=1}^{x} \ln(p_n)$$

We next look at the product of the inverse powers of p_n. One has-

$$H(x) = \prod_{n=1}^{x} \frac{1}{p_n} = (\frac{1}{2})(\frac{1}{3})(\frac{1}{5})(...)(\frac{1}{p_x}) = \frac{1}{G(x)}$$

This function goes to zero as x gets large. Its values are given by simply taking the reciprocal of G(x). We have H(10)=1/6469693230=1.54756797..x10^{-8}.

Finally let us see what happens when we look at the original function F(s,x) given above when s=σ+iτ becomes complex. In this case we have the complex function-

$$F(\sigma,\tau,x) = \sum_{n=1}^{x} \frac{\cos[\tau\ln(p_n)] - i\sin[\tau\ln(p_n)]}{p_n^{\ \sigma}}$$

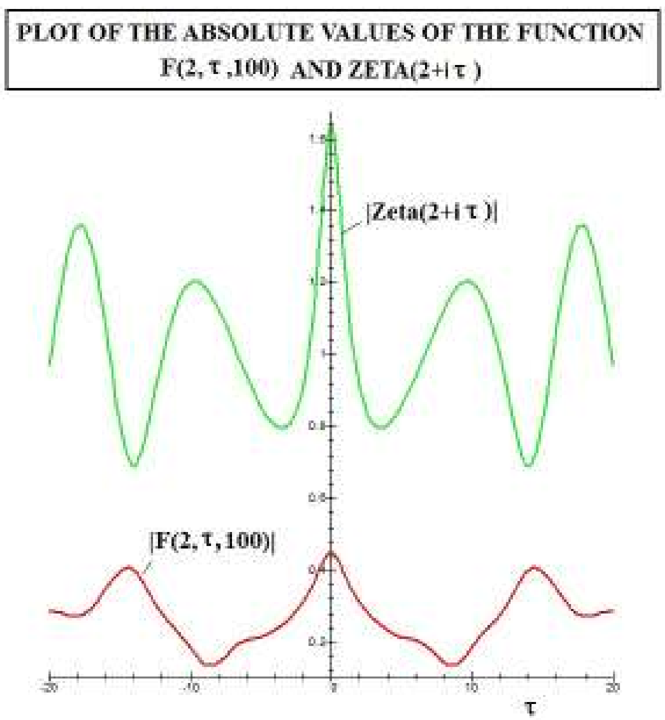
whose absolute value is-

$$|F(\sigma,\tau,x)| = \sqrt{\left[\sum_{n=1}^{x} \frac{\cos[\tau \ln(p_n)]}{p_n^{\sigma}}\right]^2 + \left[\sum_{n=1}^{x} \frac{\sin[\tau \ln(p_n)]}{p_n^{\sigma}}\right]^2}$$

Note that, for a fixed σ the function |F(σ,τ,x)| is symmetric with respect to τ and is expected to have a somewhat lower value than the corresponding Zeta Function defined by –

$$\zeta(\sigma + i\tau) = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\sigma}}\right)\left(\cos[\tau \ln(n)] - i \sin[\tau \ln(n)]\right)$$

We show you here a comparison of |F(2,τ,100)| with |ζ(2+iτ)|-



The function F(s,100) also can have zeros when s is complex. One of these zeros lies near s=0.788 +i 8.719 when using the first hundred primes. Its location shifts slightly when the first 200 prime terms are used.

The function F(s,∞) when using all primes is known in the literature as the Prime Zeta Function and has been investigated earlier by Glaisher(1891) and others.

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