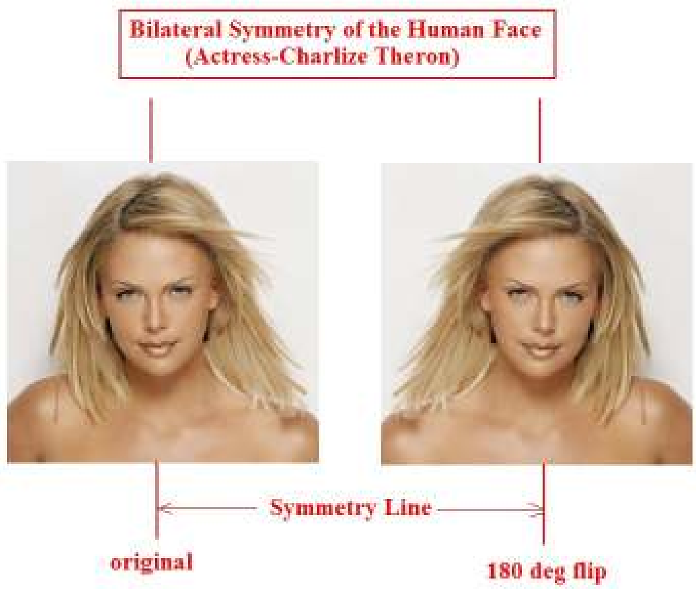
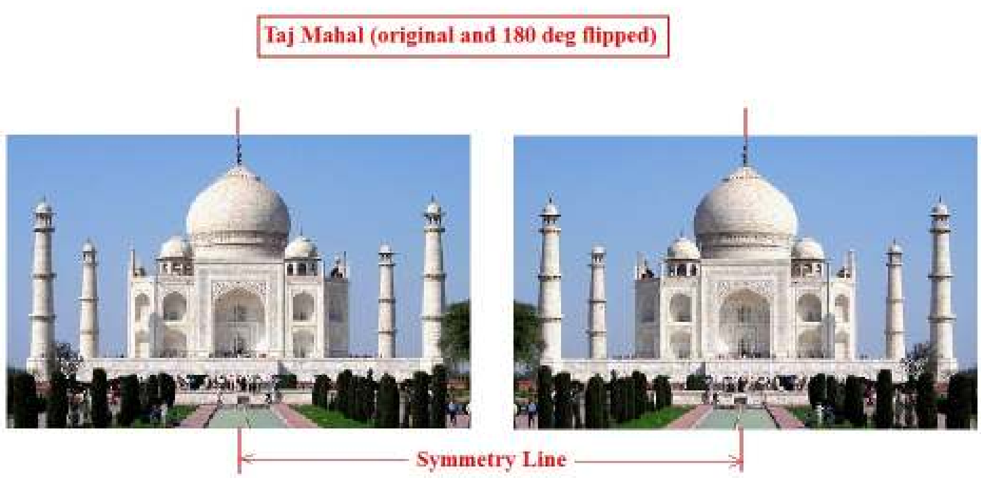
SYMMETRY

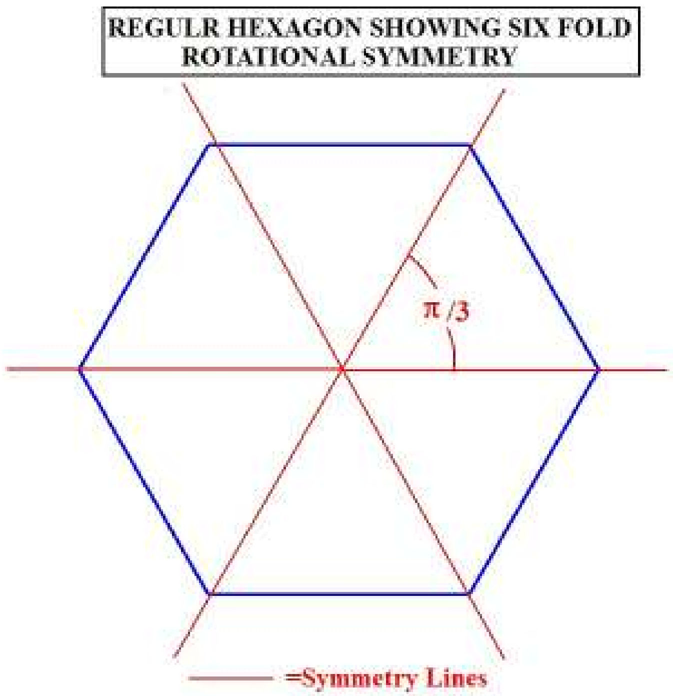
What is meant by symmetry and where is it encountered? Symmetry can be defined as that property of an object which leaves it unchanged in appearance when it is either flipped about a line or plane or rotated by a fixed angle about a point. Good examples of 2D symmetry about a line (Bilateral Symmetry) is the frontal projection of the human face and 2D projections of certain architectural structures. Here are two specific examples-





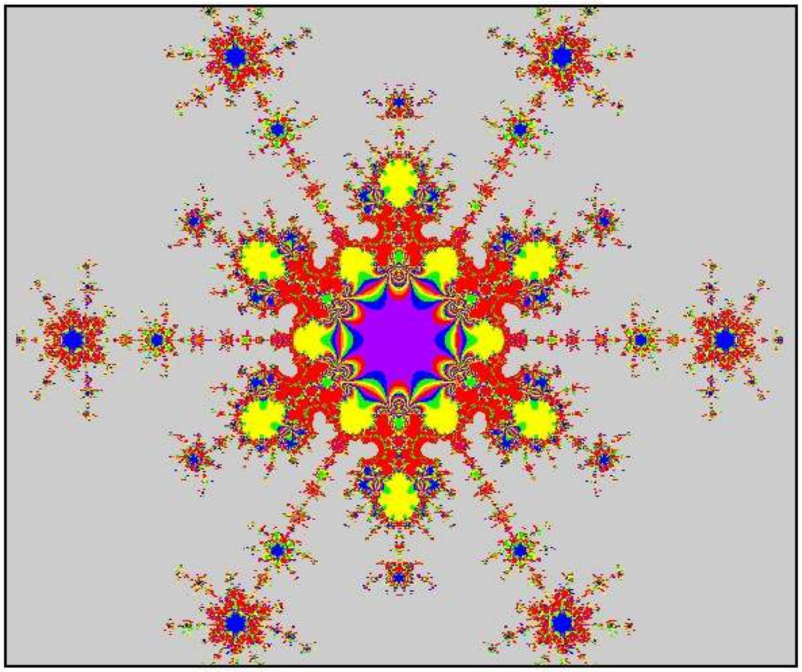
The red line represents the symmetry axis. The right image was generated from the left image by flipping the original picture horizontally by 180 degree. For a perfectly symmetric case there should be no difference in what is seen between the original and its mirror image.

Next we look at Rotational Symmetry. An object having this property will remain unchanged when rotated about a point by 2π/n radians. A good example is a regular hexagon. It has a six-fold rotational symmetry with its shape remaining unchanged when rotated by increments of π/3 radians=60deg. After any such rotation the figure remains the same as shown-



Note also that this figure has three symmetry lines. Another example of six-fold rotational symmetry arises in connection with the fractal pattern generated by the formula Z=Z^6-1.12. It produces the surprisingly intricate figure-

SIX-FOLD ROTATIONAL SYMMETRY PRODUCED BY THE FRACTAL $Z=Z^6-1.12$



Mathematically one can recognize symmetry in a function f(x,y,z) if it is invariant when changing the sign of one or more of the variables x, y, and z.. Thus the circle $x^2+y^2=1$ will not change its form when replacing x by –x and/or y by –y. One can say that it is bilaterally symmetric about the x axis and the y axis. Furthermore by introducing the rotational transformation-

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

,which rotates the circle by α radians counterclockwise, one finds –

$$(x')^2 + (y')^2 = (x\cos\alpha + y\sin\alpha)^2 + (-x\sin\alpha + y\cos\alpha)^2 = x^2 + y^2$$

Thus the circle is invariant under all angle rotations and exhibits an infinite-fold rotational symmetry. It also possesses an infinite number of symmetry lines through its center.

In 3D the symmetry lines are replaced by symmetry planes. Looking at an ellipsoid whose standard formula reads-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad with \quad a,b,and\ c\ const.$$

This solid clearly has three symmetry planes x=0, y=0, and z=0. It is also two-fold rotationally symmetric about each of the three axis x, y, and z. When a=b=c the ellipsoid degenerates into a sphere and the number of symmetry planes and rotation symmetry axes become infinite.

A cube with $-1<x<1,-1<y<1,-1<z<1$ centered on the origin has three symmetry planes x=0, y=0, and z=0.It is also four-fold rotationally symmetric about each of the three axis. In addition, when rotated about each of its four diagonal axes which pass through the cube center and opposite corners, one observes a three-fold symmetry about each of these diagonals.

Symmetry in solutions to differential equations are also found quite often. Consider a mass m hung from a spring of spring constant k. Its vertical displacement x as a function of time t is given by-

$$m\frac{d^2x}{dt^2}=-kx \text{ or the equivalent } \left(\frac{dx}{dt}\right)^2+\frac{k}{m}x^2=C$$

The second form represents essentially an ellipse of semi-major axis a=sqrtC and semi-minor axis b=sqrt(mC/k). Such a figure is symmetric about both the symmetry line x=0 and also the symmetry line dx/dt=0. It represents simple harmonic motion as viewed in the phase plane. Note that introducing a damping term of the form αdx/dt into the above second order differential equation will destroy the symmetry present in the solution.

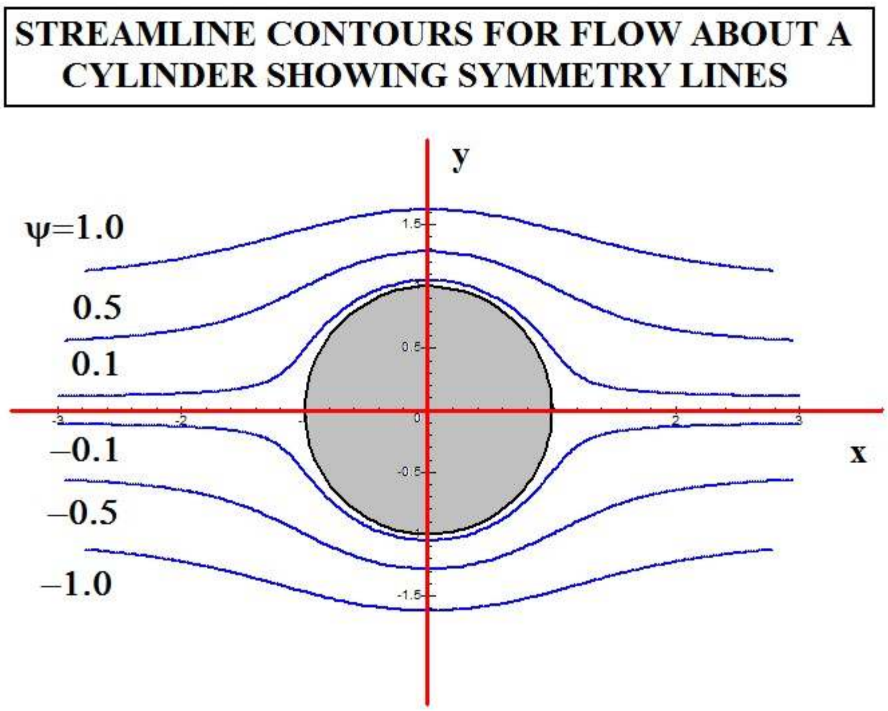
The famous Laplace partial differential equation in 2D reads-

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

where ψ represents the streamfunction (or also the velocity potential φ) when dealing with inviscid and incompressible fluid flow. For flow about a unit radius cylinder, where the streamfunction is zero on the cylinder surface, the streamfunction is symmetric about the y axis and anti-symmetric about the x axis. The explicit solutions found for the streamfunction and velocity potential are-

$$\psi = (r - \frac{1}{r})\sin(\theta) \quad and \quad \varphi = (r + \frac{1}{r})\cos(\theta)$$

where $r=\text{sqrt}(x^2+y^2)$ and $\theta=\arctan(y/x)$. The following plot indicates the streamline pattern ψ=Const.-



Note that

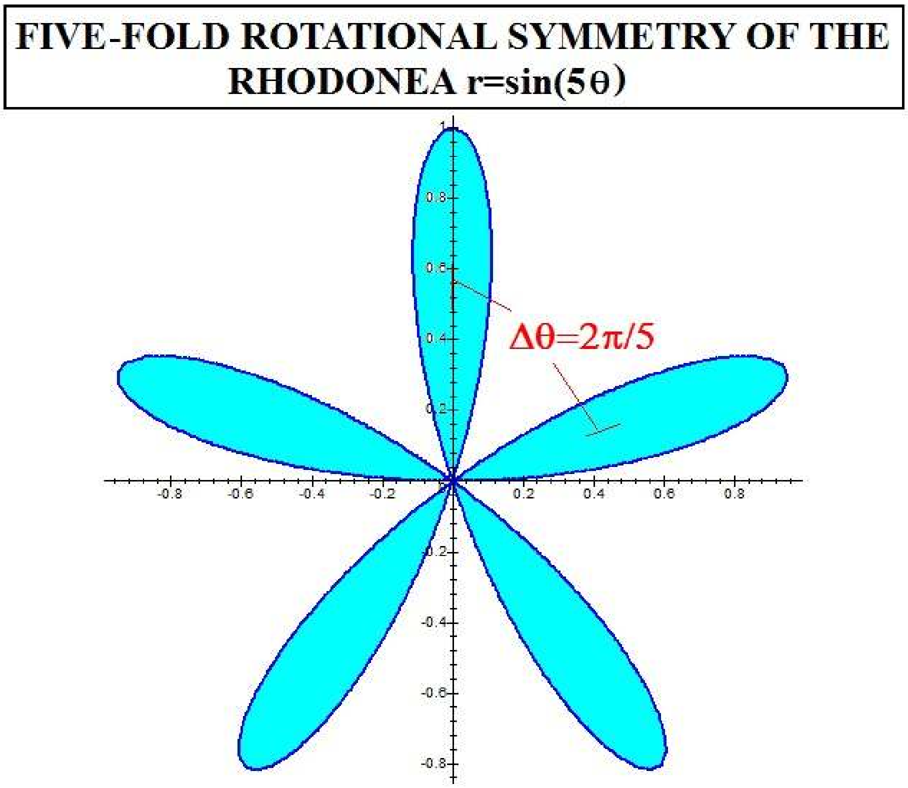
$$\psi(x,y) = \left[1 - \frac{1}{(x^2+y^2)}\right]y$$

in Cartesian coordinates. We thus have that $\psi(x,y)= +\psi(-x,y)$ but $\psi(x,y)= -\psi(x,-y)$.

Finally lets look at the symmetry of the five petal Rhodonea. It is defined by the formula-

$$r = \sin(5\theta) \; or \; (x^2+y^2)^3 = y\{(x^2+y^2)^2 - 12y^2(x^2+y^2) + 16y^4\}$$

and looks like the blades of a typical ceiling fan as shown-



It clearly has a rotationally symmetric pattern repeating every $2\pi/5$ radians. That is, the figure has a five-fold rotational symmetry. Furthermore, as seen from the Cartesian expression, the figure is unchanged by replacing x by –x but does change when replacing y by –y. Hence the y axis is a line of symmetry reminiscent of that for the human body. In addition there are four additional lines of symmetry. These lie along the radial lines passing through the center of each of the four remaining petals.