

**SEQUENCE GENERATED BY THE ITERATION  
 $A[n+1]=(a+bI)^{A[n]}$  WITH  $A[0]=a+bI$**

Consider the sequence- 2, 4, 16, 65536, etc... . How is this divergent sequence formed? A brief inspection shows it to be generated by the iteration-

$$A_{n+1} = 2^{A_n} = \exp\{A_n \ln(2)\} \quad \text{subject to } A_0 = 2$$

Such an iteration procedure is generally referred to as **Tetration**(see <http://en.wikipedia.org/wiki/Tetration>). It represents a non-associative procedure where-

$$A_{n+3} = 2^{(A_{n+2})} = 2^{(2^{A_{n+1}})} = 2^{(2^{(2^{A_n})})}$$

which is not equal to the associate form-

$$A[n + 1] = A[n]^2$$

where the sequence is 2, 4, 16, 256, etc.

By changing 2 to a complex number  $N=a+Ib$  one obtains the generalized tetration(alias power tower) form-

$$A_{n+1} = N^{A_n} \quad \text{subject to } A_0 = N = a + ib$$

which may or may not converge. It is straight forward to computerize this tetration procedure for a given value of N. Take the case  $N=I = A[0]$ . Here one has the one line MAPLE program-

**$A[0]:=I$ ; for n from 0 to 29 do  $A[n+1]:=evalf(I^{(A[n])}$  od;**

which produces-

```
A[0] := I
A[1] := .2078795764
A[2] := .9471589980 + .3207644501 I
A[3] := .05009223617 + .6021165269 I
A[4] := .3871661812 + .03052708165 I
A[5] := .7822756823 + .5446065578 I
A[6] := .1425618232 + .4004665252 I
A[7] := .5197863965 + .1183841965 I
A[8] := .5685886171 + .6050784068 I
A[9] := .2423652469 + .3011505920 I
A[10] := .5784886834 + .2315297354 I
```

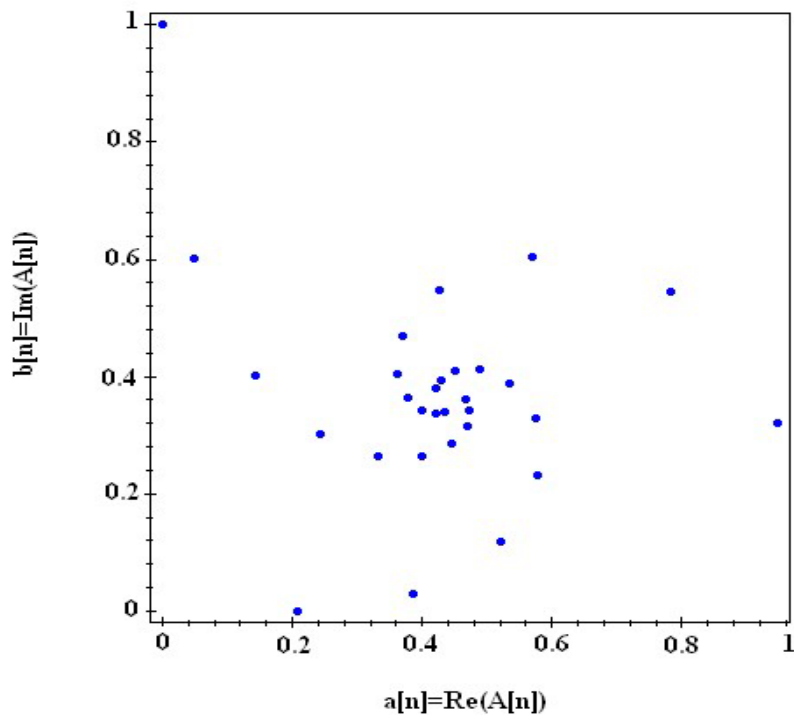
A[11] := .4273401326 + .5482312173 I  
A[12] := .3309671044 + .2628918428 I  
A[13] := .5742710154 + .3287156237 I  
A[14] := .3699480421 + .4681731933 I  
A[15] := .4006334949 + .2631202133 I  
A[16] := .5347438921 + .3893277733 I  
A[17] := .3621137377 + .4039644570 I  
A[18] := .4466968751 + .2855673283 I  
A[19] := .4876961476 + .4121748681 I  
A[20] := .3771703569 + .3628660069 I  
A[21] := .4691487095 + .3157939330 I  
A[22] := .4509329779 + .4092165301 I  
A[23] := .3993359745 + .3420785475 I  
A[24] := .4730689674 + .3429513940 I  
A[25] := .4296783599 + .3947803265 I  
A[26] := .4199478941 + .3360934948 I  
A[27] := .4660804224 + .3614679378 I  
A[28] := .4215451228 + .3788586122 I  
A[29] := .4349504313 + .3390761530 I  
A[30] := .4552953379 + .3706097379 I

What is noticed at once is that all the numbers in the sequence are complex and that as  $n$  gets large, the number  $A[n]$  appears to converge to a unique point in the complex plane. We show the location of these values in the sequence via the point plot program-

```
with(plots); pointplot({seq([Re(A[n]),Im(A[n])],n=0..29)},symbol=circle,  
color=blue);
```

The result is the three legged spiral pattern indicated-

SEQUENCE GENERATED BY  $A[n+1]=I^{A[n]}$  SUBJECT TO  $A[0]=I$



To find the point in the complex plane to which  $A[n]$  converges, we note that  $A[n]=A[n+1]$  as  $n \rightarrow \infty$ . That is –

$$A[\infty] = \exp\{A[\infty]\ln(I)\} = \exp\{-W(z)\}$$

where  $W(z) = A[\infty]\ln(1/I)$  is a complex function. Rewriting this last expression one has-

$$\ln\left(\frac{1}{I}\right) = z = W(z) \exp W(z)$$

which happens to represent the definition of the Lambert Function  $W(z)$  with  $z = \ln(1/I)$ . Evaluating things via MAPLE, one finds the ten place accurate result-

$$A[\infty] = \frac{W(\ln(\frac{1}{I}))}{\ln(\frac{1}{I})} = a[\infty] + I b[\infty] = 0.4382829367 + I 0.3605924718$$

**Different values of  $N=a+Ib$  will yield different limiting points. For example if  $N=A[0]=1/4$  one finds that the sequence converges to  $A[\infty]=1/2$ . On the other hand,  $N=A[0]=2$  leads to a divergent sequence 4, 16, 65536,....In most instances we find , if the sequence has a limiting value as  $n$  approaches infinity, that this is given by-**

$$A[\infty] = - \frac{W\{-\ln(N)\}}{\ln(N)}$$

**With  $W(z)$  being the standard Lambert Function. One arrives at this result as follows. Let  $n$  be very large and assume convergence to a value  $A[\infty]=A$ . This means-**

$$A = \exp A \ln(N) = \exp - W(z) \quad \text{say}$$

**From this expression one infers that-**

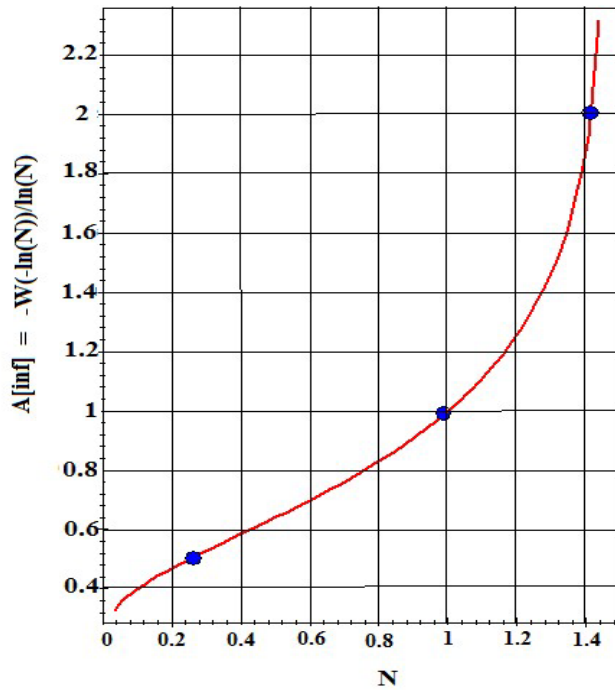
$$A = - \frac{W(z)}{\ln(N)} = \frac{-W(-\ln(N))}{\ln(N)}$$

**so that the above result is confirmed. We can now quickly check for convergence of a given sequence  $A[1], A[2], A[3], \dots A[\infty]$  using this Lambert number quotient. Take the case  $N=A[0]=\text{sqrt}(2)$ . It predicts the finite real value  $A[\infty]=2$ . The case  $N=A[0]=1$  has the obvious value  $A[\infty]=1$  and  $N=A[0]=1/4$  is found to yield the value  $A[\infty]=1/2$ . One finds that the sequence converges to a finite value when-**

$$N < 1.444667861337.. = \exp\{(1/\exp(1))\}$$

**A graph of the convergence point follows-**

CONVERGENCE VALUE FOR  $A[n+1]=N^{A[n]}$  WHEN  $N=A[0]$



In the graph we indicate as blue dots the points [0.25, 0.5], [1, 1], and [sqrt(2), 2]. The sequence will diverge when  $N > 1.44\dots$ . Our Lambert function ratio indicates a divergent value for  $A[\infty]$  when the Lambert ratio first becomes complex. Under such divergent conditions it is better to look directly at the developing sequence and see that the terms in the sequence increase rapidly with increasing  $n$ . Here are two samples of tetration on opposite sides of  $N=1.44466786\dots$ .

We find at  $N=3/2$  that-

$$\left[ \frac{3}{2} \right] \left[ \frac{3}{2} \right] \left[ \frac{3}{2} \right] \left[ \frac{3}{2} \right] \dots \rightarrow \infty \quad \text{but} \quad A[\infty] = 2.30\dots + 11.08\dots$$

while at  $N=7/5$  one finds-

$$\left[ \frac{7}{5} \right] \left[ \frac{7}{5} \right] \left[ \frac{7}{5} \right] \left[ \frac{7}{5} \right] \dots = 1.8866633\dots = A[\infty]$$

The break between divergence and convergence is typically not found when  $N$  is complex.  $N=A[0]=1+I$  yields an expected convergent value-

$$(1 + I)^{(1+I)^{(1+I)^{\dots}}} = A[\infty] = 0.641026..+I 0.523628..$$

However, at other times the complex value for  $N=A[0]$  might be expected to produce divergence of the sequence while in reality it very slowly approaches convergence. A prime example of this behavior is observed for  $N=A[0]=2+I$ . One finds that-

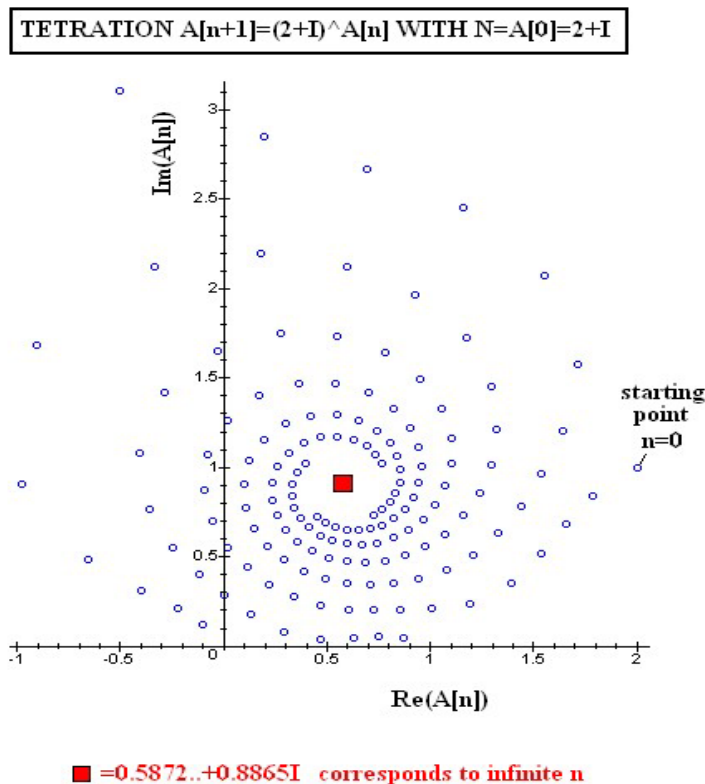
$$A_{n+1} = (2 + I)^{A_n} \quad \text{produces} \quad A_{148} = 0.4549.. + I0.7211..,$$

$$A_{149} = 0.7255.. + I0.7341.., \text{ and } A_{150} = 0.7654.. + I1.0204..$$

while the Lambert function ratio indicates eventual convergence to-

$$A[\infty] = 0.5872824194 + 0.8865950481 I$$

A point graph for this slow approach to a convergence is shown here-



**One clearly sees that the  $A[n]$  sequence approaches the convergence point predicted by the Lambert ratio value for  $A[\infty]$  but it will take hundreds of iterations to get close.**

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