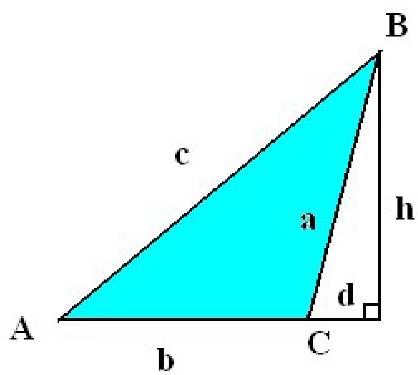
DERIVATION OF THE BASIC LAWS FOR OBLIQUE TRIANGLES

In elementary high school trigonometry one learned the basic laws for oblique triangles including the law of sines, law of cosines, Mollweide formulas, and Heron's Formula. The latter where usually just stated without proof since the mathematics is somewhat involved. Let us quickly prove all these formulas since they are very handy in a variety of areas including statics, dynamics, triangulation and surveying. Our starting point is the blue triangle shown.



By Pythagoras we have that $a^2 = h^2 + d^2 \; and \; (b+d)^2 + h^2 = c^2$

Also $\cos(\pi - C) = d/a$ **. Eliminating h and d from these three equations produces the Law of Cosines-**

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

which relates the three sides a,b,c of the oblique triangle to the cosine of the angle C opposite to side c. Next we have from the figure that sin(A)=h/c and sin(π-C)=h/a. Again eliminating h we obtain the Law of Sines-

$$\frac{\sin(A)}{a} = \frac{\sin(C)}{c} = \frac{\sin(B)}{b}$$

with the last ratio deduced from symmetry. Combining the law of Sines and Cosines one finds that-

$$\sin(A) = \frac{1}{2bc}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}$$

Thus if a=4,b=2, and c=5 , we find $\sin(A)=\frac{\sqrt{231}}{20}$ $or\ A=0.8632rad=49.46\deg$ **.We look next at some more complicated formulas, namely, those of Mollweide. Karl Mollweide(1774-1825) was an astronomer and mathematician at the University of Leipzig and came up with his formulas about 1808 although versions of them were already known earlier to Newton and some Italian mathematicians. Starting with the Law of Sines one has-**

$$\frac{a+b}{c}=\frac{\sin(A)+\sin(B)}{\sin(C)}$$

Also we know from the half angle formulas that-

$$\sin(A)+\sin(B)=2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})\ ,\ \sin(C)=2\sin(\frac{C}{2})\cos(\frac{C}{2})\ and$$

$$\cos(\frac{C}{2})=\cos(\frac{\pi-A-B}{2})=\sin(\frac{A+B}{2})$$

Thus $\frac{a+b}{c}=\frac{\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})\sin(\frac{A+B}{2})}$ **which produces the First Mollweide Formula-**

$$\frac{a+b}{c}=\frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})}$$

In a similar manner using the identity $\sin(A)-\sin(B)=2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})$ **one finds the Second Mollweide Formula-**

$$\frac{a-b}{c}=\frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}$$

We see that these last two formulas contain all sides a,b,c and all angles A,B,C. Thus they provide a convenient check for triangle calculations. Their practical value is , however, considerably less than the Law of Sines and Cosines.

Finally consider the area of an oblique triangle. The vector form of 'a' in the above figure is ia and that of c is ic cos(A)+jc sin(A). So taking the cross product, the area of the triangle becomes-

$$Area = \frac{1}{2} ac \sin(A) = \frac{1}{4}\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}$$

by using the earlier given value for sin(A). One sees at once that the area of a right triangle is just 0.5ab. Furthermore, on expressing things as a function of the triangle half perimeter s=(a+b+c)/2, this result can also be written as Heron's Formula-

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

The polymath Heron of Alexandria(approx. 10-70 AD)gave a rather complicated geometrical proof of this formula. It may not be original with him since many of his text references were to earlier Babylonian mathematicians. It is known, however, that he invented the aeolipile, a steam driven rotational device and hence can be considered the father of the steam engine.

One can use the area formula for triangles to evaluate the area of an n sided regular polygon of side length c. One can construct n sub-triangles having the sides c and a=b=c/[2 cos(π/2-π/n)]. The area of this sub-triangle is (c^2/4)cot(π/n),so that the area of the polygon becomes-

$$Area = \frac{c^2 n}{4\tan(\frac{\pi}{n})}$$

For an octagon of sides c=1 this area is 2cot(π/8)=2[1+sqrt(2)]=4.828427… . Note also for n approaching infinity the sides a and b of the sub-triangle approach the radius of a circle. Thus one has-

$$\pi = \lim_{n \to \infty} n \sin(\frac{\pi}{n})$$

which forms the basis for Archimedes's method for finding the value of π. For a 100,000 sided polygon one finds the above ratio to be 3.14159265307… compared to π= 3.141592653589…. So pretty close, but clearly a losing approach when interested in highly accurate values of π.