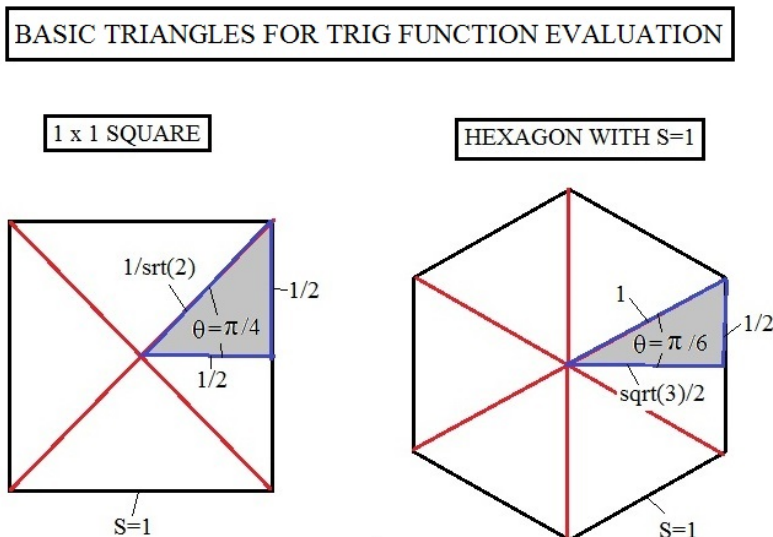


EVALUATION OF TRIGONOMETRIC FUNCTIONS

When first exposed to trigonometric functions in high school, students are expected to memorize the values of the trigonometric functions of sine, cosine, and tangent for the special cases of 45° and 30° and 60° without much discussion as to how these values and related ones come about. We want here to review how these values are established and show how they may be used in conjunction with various trigonometric identities to find the exact trigonometric values for other angles expressed in terms of various combinations of integers.

Our starting points are a standard regular square and hexagon of side-length $S=1$ each. By drawing dividing lines across the interior of these figures one can construct two basic right triangles shown in grey in the following figure-



From the geometry one sees that the first right triangle has angles of $\pi/4$, $\pi/2$, and $\pi/4$ when expressed in radians. Also by the Pythagorean theorem one finds the hypotenuse of the first triangle to be $1/\sqrt{2}$. The second right triangle shown has the angles $\pi/6$, $\pi/2$, and $\pi/3$ with a hypotenuse of length 1. Next, using the definition of the tangent function applied to these two triangles, we have-

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad , \quad \tan\left(\frac{\pi}{4}\right) = 1 \quad \text{and} \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

For the sine function and cosine function, we get-

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad , \quad \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad , \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

If one then makes use of the double angle formula for tangent and inverts it, one has the identity-

$$\tan\left(\frac{A}{2}\right) = \frac{[\sqrt{1 + \tan(A)^2} - 1]}{\tan(A)}$$

Also we have that-

$$\sin(A) = \frac{\tan(A)}{\sqrt{1 + \tan(A)^2}} \quad \text{and} \quad \cos(A) = \frac{1}{\sqrt{1 + \tan(A)^2}}$$

These last three expressions may now be used to calculate the tangent, sine and cosine of all angles equal to a fraction $1/2^n$ of the original angles of $\pi/3$, $\pi/4$, and $\pi/6$. We find-

$$\tan\left(\frac{\pi}{8}\right) = [\sqrt{2} - 1] \quad , \quad \sin\left(\frac{\pi}{8}\right) = \left[\frac{\sqrt{2 - \sqrt{2}}}{2}\right] \quad \text{and} \quad \cos\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{4 - 2\sqrt{2}}}$$

and-

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} \quad , \quad \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2 - \sqrt{3}}}{2} \quad \text{and} \quad \cos\left(\frac{\pi}{12}\right) = \frac{1}{2\sqrt{2 - \sqrt{3}}}$$

Combining $\tan(\pi/8)$ and $\tan(\pi/12)$, one also has-

$$\tan\left(\frac{\pi}{8} - \frac{\pi}{12}\right) = \tan\left(\frac{\pi}{24}\right) = \frac{\tan\left(\frac{\pi}{8}\right) - \tan\left(\frac{\pi}{12}\right)}{1 + \tan\left(\frac{\pi}{8}\right)\tan\left(\frac{\pi}{12}\right)} = \frac{(\sqrt{2} + \sqrt{3} - 3)}{1 + (\sqrt{2} - 1)(2 - \sqrt{3})}$$

and-

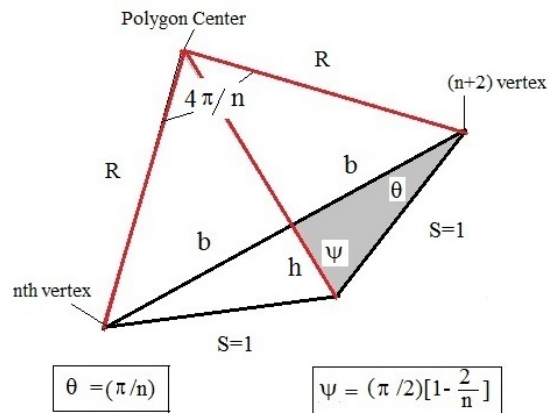
$$\tan\left(\frac{\pi}{16}\right) = \frac{\sqrt{1 + \tan\left(\frac{\pi}{8}\right)^2}}{\tan\left(\frac{\pi}{8}\right)} = \frac{[\sqrt{4 - 2\sqrt{2}} - 1]}{(\sqrt{2} - 1)}$$

The above approach will yield exact values for all trigonometric functions at angles-

$$\theta = \frac{\pi}{3 \cdot 2^n} \text{ and } \frac{\pi}{4 \cdot 2^n} \text{ and combinations thereof}$$

To generate the function values for other angles one needs to change the initial grey right triangles shown in the above figure. If we are looking at trigonometric values for angles of the form $\pi/(n \cdot 2^k)$ one should start with a regular n sided polygon of side length $S=1$. The base triangle will here be formed from the isosceles triangle formed by drawing a straight line between vertex n and $n+2$ as shown-

BASE TRIANGLE CONSTRUCTED FROM n SIDED POLYGON



The base triangle shown in grey equals exactly half of larger isosceles triangle. The gray right triangle has acute angles θ and ψ . From the geometry one sees that-

$$\psi = \left(\frac{\pi}{2}\right)\left[1 - \frac{2}{n}\right] \quad \text{and} \quad \theta = \left(\frac{\pi}{n}\right)$$

Also applying the Law of Cosines and the Law of Sines to the isosceles triangle, we find-

$$c = 2b = 2 \cos\left(\frac{\pi}{n}\right) = \frac{\sin\left(\frac{\pi(n-2)}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

We can look at this last result as an algebraic equation-

$$c = 2b = 2 \cos(x) = \frac{\sin[x(n-2)]}{\sin(x)} \quad \text{with} \quad x = \frac{\pi}{n}$$

Here the sine quotient can be expanded as a polynomial in powers of c once n has been given. In terms of c one needs to solve-

$$c = \sum_{k=0}^{k < (n-3)/2} (-1)^k \frac{(n-3-k)!}{k!(n-2k-3)!} c^{n-3-2k}$$

Take now the special case of n=5 corresponding to a pentagon . There n=5, $x=\pi/5$, $\psi=3\pi/10$ and $\theta=\pi/5$. So our algebraic equation reads-

$$c = c^2 - 1$$

This equation can be solved via the quadratic formula and yields the solution-

$$c = 2b = \frac{(1 + \sqrt{5})}{2} = 1.61803398\dots$$

which is the famous Golden Ratio. Thus, the length $c = 2b$ of the line extending from the nth to n+2 vertex of a regular pentagon with S=1 just equals the golden ratio.

We now know all the side-length of the base triangle for n=5. The hypotenuse is S=1, and the remaining two sides have length $b = \cos(\pi/5)$ and $h = \sin(\pi/5)$. Since b has been found, we have-

$$\cos\left(\frac{\pi}{5}\right) = b = \frac{1 + \sqrt{5}}{4}, \quad \sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{2(5 - \sqrt{5})}}{4} \quad \text{and} \quad \tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$$

Using some half-angle trigonometric formulas we can also rapidly calculate other trigonometric values such as $\cos(\pi/10)$ which may be written as-

$$\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{1 + \cos\left(\frac{\pi}{5}\right)}{2}} = \left(\frac{1}{2}\right) \sqrt{\frac{5 + \sqrt{5}}{2}}$$

An interesting observation about the $\pi/10$ angle is that the ten-sided regular polygon used to construct its base triangle has sides S=1 and the distance from any of its vertices to the center is exactly equal to the golden ratio $\phi = [1 + \sqrt{5}]/2$.

It is clear from the above that one can find trigonometric values for any angle π/N , when N is an integer. One starts with a construction of the base triangle using an N sided regular polygon of sides S=1, then writes down the generic values for θ and ψ , and next solves the resultant nonlinear algebraic equation in $x = \pi/N$. Once x is found, the specific values of all sides and angles of the base triangle will be known and one can write down the trigonometric values for the angles. Double and half angle formulas are used to find

trigonometric values for angles larger or smaller than π/N by factors of 2^k . In all of the above cases we looked at polygons with $N=2, 3,$ and 5 sides or multiples of 2 thereof. The numbers $2, 3,$ and 5 represent the first three prime numbers. Every prime number N will require a new base triangle.

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