USE OF LEGENDRE POLYNOMIALS TO CONSTRUCT ACCURATE TABLES FOR THE TRIGONOMETRIC FUNCTIONS

About a decade ago while I was playing around with integrals containing Legendre Polynomials P(n,x) it became clear to me that certain integrals involving the even Legendre polynomials P(2n,x), when multiplied by certain functions f(ax) and the product integrated over the range 0<x<1, can lead to excellent approximations for certain functions g(a). In particular we found that the integral-

$$J(a,n)=\int_{x=0}^{1}\cos(ax)P(2n,x)dx$$

yields excellent approximations for tan(a) to any desired order of accuracy by making n large enough. You can find a summary of these results, written in conjunction with my colleague and co-author Sidey Timmins, by clicking on the following -

http://www2.mae.ufl.edu/~uhk/IEEETrigpaper8.pdf

This method for quickly finding approximations for certain functions by using Legendre Polynomials of high order is now referred to in the literature as the KTL Method. See-

https://wiki.tcllang.org/page/Trig+Procedures+for+degree+measures+as+sind%2C+cosd%2C+tand%2Cetc

Although I have not looked at this approximation method again for nearly a decade, in the last few months my interest has been revived especially in regard to finding additional functions f(ax) which can lead to interesting and improved approximations for certain analytic functions g(a). For my latest results in this area click on-

http://www2.mae.ufl.edu/~uhk/KTL-METHOD.pdf

It is the purpose of the present note to look further at using the particular function f(ax)=cos(ax) for larger n and thus finding approximations to trigonometric functions of higher accuracy than previously achieved. It should be noted that the KTL method places no limits on the size of n one can use so that fifty digit accurate approximations for tan(1) and hence sin(a) and cos(a) should be possible. With aide of a PC, using a mathematics program such as MAPLE, one should be able to handle the very large polynomial quotients arising for large n without having to write out these quotients by hand.

We begin our analysis by noting that the above integral J(a,n) can always be expanded as

$$J(a,n)=M(a,n)\cos(a)+N(a,n)\sin(a)$$

, where N and M are long polynomials in ‘a’ of order 2n-1 and 2n, respectively, for a given n. As n gets large the integral J(a,n) will head toward zero, leaving us with the approximation-

$$\tan(a) \approx -\frac{M(a,n)}{N(a,n)}$$

This approximation, as we shall see, gives multiple place accuracy for tan(a) with the accuracy increasing with increasing n.

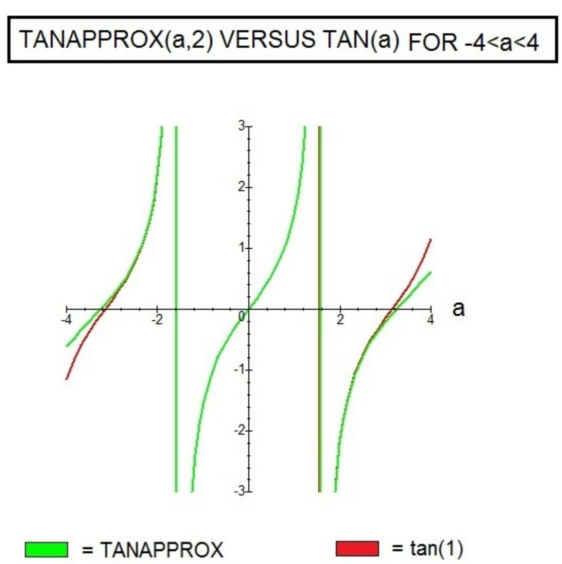
Here is the simple MAPLE computer procedure we use to find the approximation TANAPPROX(a,n)-

(1)-choose a value for n
(2)-next expand J(a,n) = int(cos(a*x)*P(2*n,x),x=0..1) to produce M(a,n)cos(a)+N(a,n)sin(a)
(3)-then use collect(J(a,n),{sin(a),cos(a)}) to separate the sin(a) from the cos(a) terms
(4)- find TANAPPROX(a,n) by doing evalf(-M(n,a)/N(n,a),k), with k being the number of digits desired.

As a first calculation we take n=2. It yields-

$$\tan(a) \approx TANAPPROX(a,2) = \frac{105a - 10a^3}{105 - 45a^2 + a^4}$$

and produces the plot-



It is amazing how, for such a low value of n, the approximation lies so close to the actual tan(a) values when |a|<2. The approximation yields 1.557377..compared to tan(1)=1.557407724.. . Also it shows an infinity at a=1.571233.. compared to the exact value of a=π/2=1.570796..

To get a feel of how our TANAPPROC(a,n) approaches the value of tan(a), we carried out calculations at a=π/4 corresponding to the angle 45deg for n=2,4,6,8,10 and 12. The results are summarized in the following table-

n	tan(π/4)-TANAPPROX(π/4,n)
2	$0.21312\text{x}10^{-5}$
4	$0.45429\text{x}10^{-15}$
6	$0.18600\text{x}10^{-26}$
8	$0.55742\text{x}10^{-39}$
10	$0.23471\text{x}10^{-52}$
12	$0.20501\text{x}10^{-66}$

We see that the accuracy of the tan(a) approximation for an angle of 45deg (equivalent to a=π/4) goes up about five decimal places per unit increase in n. So we estimate an approximation for tan(π/4) will be accurate to100 decimal places when n=18.

To construct an accurate trigonometric table (or computer subroutine) good to 50 decimal places over the range 0<a<π/4 should be possible using TANAPPROX(a,10). One knows from elementary trigonometry that if tan(x) is known between 0 and π/4 all values for tan(a) outside this range will also be known. This follows from the identity-

$$\tan\left(\frac{\pi}{4}-b\right)=\frac{1}{\tan\left(\frac{\pi}{4}+b\right)}\quad with\quad 0<b<\frac{\pi}{4}$$

It is also known that-

$$\cos(a)=\frac{1}{\sqrt{1+T^2}}\quad and\quad \sin(a)=\frac{T}{\sqrt{1+T^2}}$$

We are here using the abbreviation T=TANAPPROX(a,10) to simplify the bookkeeping..

To get the value of the elements for a trig table at every 5deg intervals over 0≤a≤π/4 we use the two line program-

a:=(π/4-kπ/36); for k from 0 to 9 do
{45*(1-k/9),evalf(T,50),evalf(1/sqrt(1+T^2),50),evalf(T/sqrt(1+T^2),50)}od;

This entire table is calculated in a split second. It being rather lengthy, I just give you the 50 digit long accurate results for 30deg (a=π/6)-

tan(π/6)= 0.57735026918962576450914878050195745564760175127012

$\cos(\pi/6)= 0.86602540378443864676372317075293618347140262690519$
$\sin(\pi/6)= 0.50000000000000000000000000000000000000000000000000$

We are aware of no other existing program which can match the speed and simplicity of the present KTL method in finding these values. When seeing something like this, my mind often wanders back to feel sorry for the WWII British mathematicians who in pre-electronic computer days spent countless hours producing logarithmic and trigonometric tables accurate to no more than 20 places of decimal.

U.H.Kurzweg
September 10, 2019
Gainesville, Florida