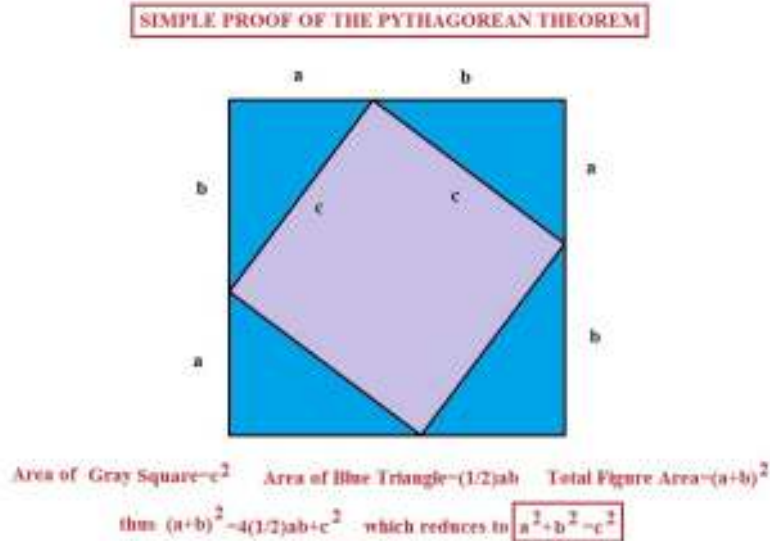


RIGHT TRIANGLES AND THE PYTHAGOREAN TRIPLETS

Known for over 2500 years is the fact that the sum of the squares of the legs of a right triangle equals the square of the hypotenuse. That is $a^2+b^2=c^2$. A simple proof is shown in the following figure -



As already shown by Euclid in his “Elements”, there are an infinite number of such right angle triangles whose sides have all integer values. They are referred to as triangle integer triplets and designated by [a,b,c]. The first few of these triplets where $a < b < c$ can be found by a simple numerical evaluation and are-

[a,b,c]=[3,4,5],[5,12,13],[6,8,10],[7,24,25],[8,15,17],[9,12,15],[9,40,41],[10,24,26],
 [11,60,61],[12,16,20],[12,35,37],[13,84,85],[14,48,50],[15,112,113],[16,63,65],
 [17,144,145],[18,80,82],[19,180,181],[20,99,101],[20,21,29]...

One notices at once that some of the triplets are just an earlier triplet multiplied by an integer and thus form similar triangles. Thus [6,8,10], [9,12,15],[12,16,20] are all similar to the [3,4,5] **base triplet** for the group. Likewise [5,12,13] is the **base triplet** for the group [10,24,26],[15,36,39], etc. Also no real new information is obtained by interchanging ‘a’ and ‘b’. Keeping only the base triplets leaves one with-

[3,4,5],[5,12,13],[7,24,25],[8,15,17],[9,40,41],[11,60,61],[12,35,37],[13,84,85],[14,48,50]
 [15,112,113],[16,63,65],[17,144,145],[18,80,82],[19,180,181],[20,99,101],[20,21,29],...

You will notice in this collection that the numbers ‘b’ and ‘c’ differ by one if ‘a’ is an odd integer and by two if ‘a’ is an even integer for most cases with a few exceptions such

as [20,21,29]. This means we may generate a subclass of the infinite number of base triplets by the formulas-

$$a = n, \quad b = (n^2 - 4)/4, \quad \text{and} \quad c = (n^2 + 4)/4 \quad \text{for even } a$$

and

$$a = n, \quad b = (n^2 - 1)/2, \quad \text{and} \quad c = (n^2 + 1)/2 \quad \text{for odd } a$$

with $n=1, 2, 3, \dots$. As examples, for $a=321$ we find $b=51520$ and $c=51521$ and for $a=424$ we have $b=44943$ and $c=44945$. Both triplets satisfy the Pythagorean Theorem for right triangles. Note that in the above list the last triplet [20,21,29] does not satisfy the just stated formula for even 'a' nor is it obtainable from one of the lower number base triplets. The difference between c and b is 8 suggesting the modification-

$$a = n, \quad b = [n^2 - \Delta^2]/2\Delta, \quad \text{and} \quad c = [n^2 + \Delta^2]/2\Delta$$

where $\Delta=(c-b)k$ and $k=1, 2, 3, \dots$ for both even and odd 'a'. We can also multiply this last triplet by 2Δ to obtain the all integer triplets-

$$a = 2\Delta n, \quad b = n^2 - \Delta^2, \quad c = n^2 + \Delta^2 \quad \text{with} \quad n > \Delta$$

This last expression represents the classical form for triplets already known to Euclid and referred to in the literature as the Pythagorean Triplets. One example of such a triplet obtained for $\Delta=99$ and $n=121$ is-

$$[a, b, c] = [23958, 4840, 24442]$$

Note that this Pythagorean Triplet is not a base triplet since all components can be divided by 242 to yield the base triplet [20,99,101]. Among the above listed base triplets we find [20,21,29] is generated by $\Delta=2$ and $n=5$. Also [20,99,101] is generated by $\Delta=1$ and $n=10$.

To determine if a triplet generated by the Euclid formula is a base triplet one needs to simply see if the elements divide by an integer. Thus-

$$[96,180,204] \rightarrow 4[24,45,51] \rightarrow 12[8,15,17]$$

shows that the base triplet in this case is [8,15,17] and is characterized by $n=4, \Delta=1$.

One can take the original Pythagorean Theorem and divide it by c^2 . Then letting $x=a/c$ and $y=b/c$, one has the formula for a unit radius circle-

$$x^2 + y^2 = \left[\frac{2n\Delta}{n^2 + \Delta^2} \right]^2 + \left[\frac{n^2 - \Delta^2}{n^2 + \Delta^2} \right]^2 = 1$$

and can read off the points on this circle corresponding to a base triplet by the appropriate choice of n and Δ . Keeping 'a' an even number, one has for the first few base triplets-

$$[n, \Delta] = [2, 1], [3, 2], [4, 3], [4, 1], [6, 5], [6, 1], [8, 7], [8, 1], [9, 8], [9, 1], [10, 9], \dots$$

The corresponding values of x and y are-

$$[x, y] = \left[\frac{4}{5}, \frac{3}{5} \right], \left[\frac{12}{13}, \frac{5}{13} \right], \left[\frac{24}{25}, \frac{7}{25} \right], \left[\frac{8}{17}, \frac{15}{17} \right], \left[\frac{60}{61}, \frac{11}{61} \right], \left[\frac{12}{37}, \frac{35}{37} \right],$$

$$\left[\frac{112}{113}, \frac{15}{113} \right], \left[\frac{16}{65}, \frac{63}{65} \right], \left[\frac{144}{145}, \frac{17}{145} \right], \left[\frac{18}{82}, \frac{80}{82} \right], \left[\frac{180}{181}, \frac{19}{181} \right], \dots$$

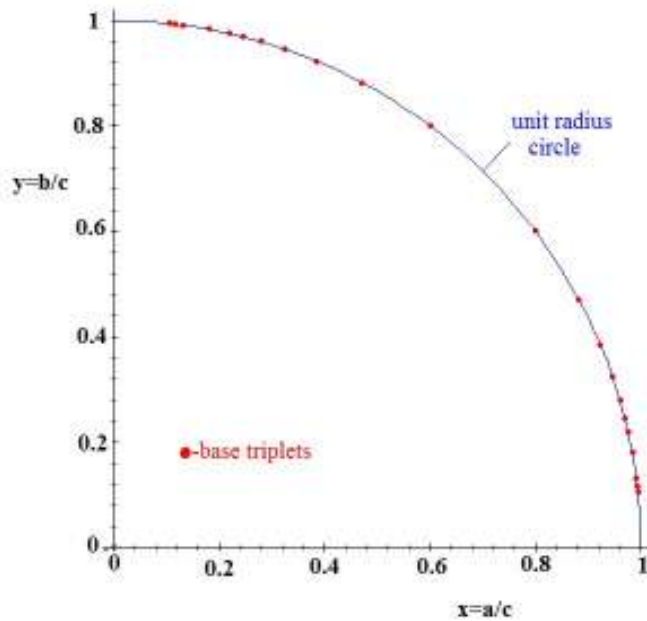
Noting that x can be interchanged with y , we have the base triplets-

$$[a, b, c] = [3, 4, 5], [5, 12, 13], [7, 24, 25], [8, 15, 17], [11, 60, 61], [[12, 35, 37],$$

$$[15, 112, 113], [16, 63, 65], [17, 144, 145], [18, 80, 81], [19, 180, 181], \dots$$

All points $[x, y]$ and their complements $[y, x]$ lie on a circle of radius one as shown-

LOCATION OF THE BASE TRIPLETS ALONG A UNIT RADIUS CIRCLE IN THE X-Y PLANE



The two acute angles of the right triangles defined by a base triplet $[a,b,c]$ are $B=\arctan(b/a)$ and $A=\arctan(a/b)$. Thus we have that-

$$\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{a}{b}\right) = \frac{\pi}{2}$$

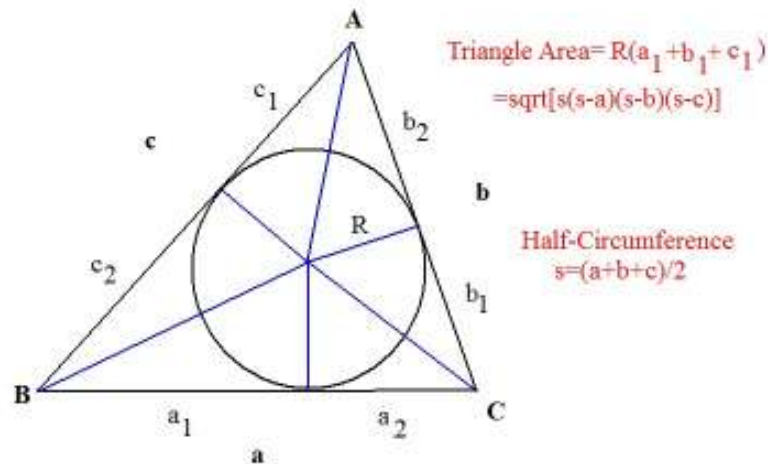
Also, it follows that-

$$\frac{a}{b} = \frac{\sin(A)}{\sin(B)} \quad \text{and} \quad \cos(A) = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos(B) = \frac{a}{\sqrt{a^2 + b^2}}$$

For the $[3,4,5]$ right triangle one has $A=\arccos(4/5)=36.869\dots\text{deg}$ and $B=\arccos(3/5)=53.130\dots\text{deg}$ with $C=90\text{deg}$.

An interesting calculation involving right triangles is the determination of the radius of the largest circle which may be inscribed in an oblique triangle. The derivation involves the Heron formula for the area of such a triangle and the bisection of the three angles A, B, C of the triangle surrounding a circle of radius R . We have the following geometry-

GEOMETRY FOR FINDING THE RADIUS OF THE LARGEST CIRCLE WHICH MAY BE INSCRIBED IN AN OBLIQUE TRIANGLE



One sees that the area of the triangle ABC must equal the area of the six right triangles shown. The height of each of the triangles is equal to the inscribed circle radius R. From the geometry one sees that the half circumference is-

$$s = \frac{(a + b + c)}{2} = \frac{(a_1 + a_2 + b_1 + b_2 + c_1 + c_2)}{2}$$

and also it is observed from the picture that-

$$a_1 = c_2, \quad a_2 = b_1, \quad b_2 = c_1$$

Thus one has after a little manipulation that-

$$a_1 = s - b, \quad b_1 = s - c, \quad c_1 = s - a \quad \text{so that} \quad s = a_1 + b_1 + c_1,$$

which means that the triangle area is-

$$\text{Area} = (a_1 + b_1 + c_1)R = sR$$

Equating to the known Heron result, one has that the radius of the largest inscribed circle in any oblique triangle will be-

$$R = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

The largest circle fitting into a [3,4,5] right triangle will thus be R=1 and for an equilateral triangle of sidelength 'a' the radius will be R=a/[2sqrt(3)]. The radius of the largest circle inscribed in the right triangle defined by the triplet [20,99,101] will be R=9. An interesting observation is that R is always an integer value as long as the triplet [a,b,c] consists of integers and defines a right triangle. This condition is violated for non right triangles such as isosceles and equilateral triangles or right triangles where one of the sides is a non-integer such as for [1,1,sqrt(2)]. Just to confirm this statement consider the base triple [1060808, 710985, 1524413] generated by n=997 and Δ=532. Here we find the largest inscribed circle has integer radius-

$$R = 247380 \quad \text{with a semi perimeter } s = 1524413$$

The fraction of the triangle area covered by the circle will be-

$$f = \frac{\text{circle area}}{\text{triangle area}} = \frac{\pi}{s^{3/2}} \sqrt{(s-a)(s-b)(s-c)}$$

Finally, we look for the largest right angle triangle, designated by the triple[a,b,c], which just fits into a circle. The solution is straight forward when one recalls from elementary geometry that two lines drawn from opposite ends of a circle's diameter always intersect at a right angle when they meet somewhere on the circle. Thus the radius of such a circle will be R=c/2 and the fraction of the circle covered by the [a,b,c] triangle will be-

$$f = \frac{\text{triangle area}}{\text{circle area}} = \frac{2ab}{\pi(a^2 + b^2)}$$

Thus a [3,4,5] right triangle will have an f=0.305577.. . This problem becomes more complicated when the triangle becomes oblique. Now the radial distance from the circle center is R and the circle touches all three corners of the [a,b,c] triangle. Calling the three angles subtended at the circle center as θ_A, θ_B,θ_C we have from the law of cosines that-

$$\cos(\theta_A) = 1 - \frac{a^2}{2R^2}, \cos(\theta_B) = 1 - \frac{b^2}{2R^2}, \cos(\theta_C) = 1 - \frac{c^2}{2R^2}$$

Denoting these cosines by α , β and γ , we have-

$$2\pi = \arccos(\alpha) + \arccos(\beta) + \arccos(\gamma)$$

Combining these three arccosine terms two at a time and then taking the cos of the result leads to the equality-

$$(\gamma - \alpha\beta)^2 = (1 - \alpha^2)(1 - \beta^2)$$

Which yields the implicit result for the circle radius R for any inscribed triangle. Take the case of an equilateral triangle of side length $a=b=c=1$. Here-

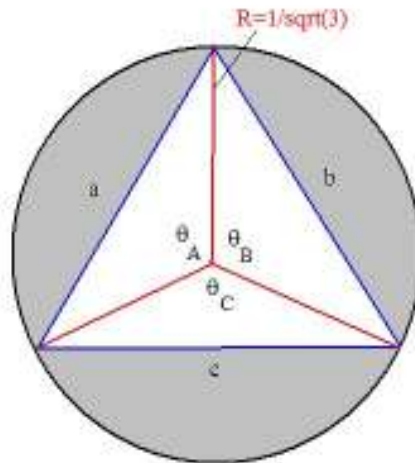
$$\alpha = \beta = \gamma = 1 - \frac{1}{2R^2}$$

And leads to the cubic equation-

$$2\gamma^3 - 3\gamma + 1 = 0$$

with solutions $\gamma = -1/2$, 1, and 1. The first of these is of interest to us and leads to $R = 1/\sqrt{3}$. A plot of this inscribed equilateral triangle follows-

LARGEST AREA TRIANGLE WHICH CAN BE INSCRIBED IN A CIRCLE



$$f = 3\sqrt{3}(4\pi) = 0.413496671..$$

Its cover fraction equals $f=3\sqrt{3}/(4\pi)=0.413496671\dots$ and is the largest possible for an inscribed triangle within a circle.

September 2010