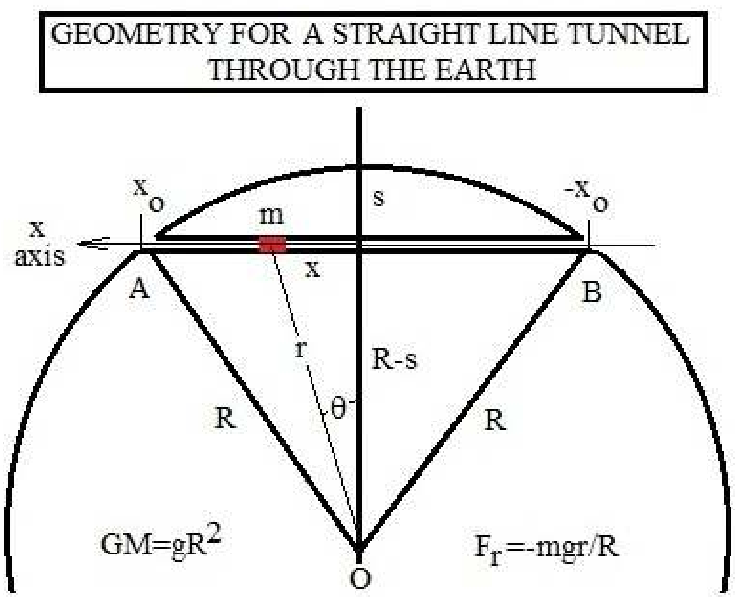
# CHARACTERISTICS OF A SRAIGHT-LINE TUNNEL DUG THROUGH THE EARTH

One of the classic elementary mechanics problems one learns about in introductory physics class is the behavior of a test mass m when dropped through an imaginary shaft extending through the center of the earth. It is shown that such a mass undergoes simple harmonic motion. If one extends the discussion to off-center shafts passing between point A and point B on the surface of the earth one is naturally led to the concept of a straight line tunnel as shown in the following figure-




The tunnel extends along the x axis from $+x_o$ to $-x_o$. A mass m started from rest at $x=x_o$ will move along the tunnel with ever increasing speed until x=0 and then will decelerate until it comes to rest at position $x=-x_o$. This assumes that there is negligible friction present. The force on the mass is strictly radial toward the earth center and is given by the law of universal gravitation as-

$$F_r = -\frac{GmM_r}{r^2} = -mg(\frac{r}{R})$$

In obtaining the second form for this force we have set $GM=gR^2$ and assumed a uniform density earth such that the mass $M_r$ lying below r relates to the total earth mass M as $M_r/M=(r/R)^3$. We have made use of the fact that the attractive force at radial distance r inside a uniform density sphere with radius R involves only the mass lying below r. If we now formulate the equation of motion for the mass m in the tunnel, we get-

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - \frac{mg}{R}x$$

Neglecting the friction term by setting β=0 and applying the initial conditions that $x(0)=x_0$ and dx(0)/dt=0, one gets the very simple solution-

$$x(t) = x_0\cos(\sqrt{\frac{g}{R}}t)$$

This result shows that the mass m is undergoing simple harmonic motion with the same angular frequency ω=(2π/τ)=sqrt(mg/R) as a mass falling through a shaft passing through the center of the earth. It predicts a travel time through the tunnel from $x=x_0$ to $x= -x_o$ of-

$$\frac{\tau}{2} = \pi\sqrt{\frac{R}{g}} = \pi\sqrt{\frac{3960\cdot 5280}{32}} = 2539\sec$$

That is, it will take about 42 minutes to go from one end of the tunnel to the other. It does not matter how long the tunnel is.

We can use the conservation of energy statement-

$$E = \frac{mg}{2R}{x_0}^2 = (\frac{1}{2})m(\frac{dx}{dt})^2 + \frac{mg}{2R}x^2$$

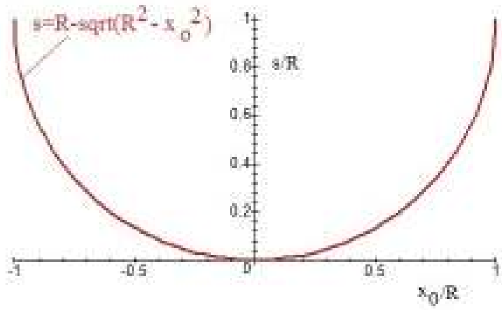
To show that the maximum speed occurs half way though the tunnel and equals-

$$(\frac{dx}{dt})_{\max} = \sqrt{\frac{g}{R}}x_0 = \sqrt{\frac{g}{R}[s(2R-s)]}$$

, where s represents the Sagitta to the chord A-B as shown. Should the Sagitta become equal to R then we are dealing with a tunnel passing through the center of the earth and the speed maximum becomes $(dx/dt)_{max}$=sqrt(gR) which happens to be equal to the speed of a near earth satellite in circular orbit.

A straight-line tunnel built between New York City and Washington DC would have a length of 204 miles and a Sagitta value of s=R-sqrt($R^2$-${x_0}^2$)=1.26 miles. The top speed reached by mass m would equal 666.26ft/sec=454 mph somewhere under Philadelphia. A plot of tunnel length $2x_0$ versus Sagitta follows-

TUNNEL LENGTH $2x_o$ VERSUS SAGITTA s




approximation for small $x_o$ is $x_o$ =sqrt(2Rs)

Note that tunnels of much more than about 500 mile length become impractical because of the large Sagitta and hence depth required.

The actual construction of such a tunnel would require no right of way and the tunnel would provide a very inexpensive way for moving freight. The construction would, however, encounter problems of high temperature, high pressure, and very high initial cost. One knows from data on South African gold mines that temperatures of 60 deg C are encountered at 12,000 ft below the surface and very high hydraulic pressures of $p=\rho gD$ are also found at larger depths D for a crust of density $\rho$. From Swiss construction data one knows that the new St. Gotthard Railway Tunnel to be completed in a few years has overcome temperature and pressure problems for overburdens as high as 8200 ft. The cost of the 34.5 mile long tunnel (which will be the world's longest ) is presently estimated to cost about ten billion US dollars equivalent. By a straight forward extrapolation for the 204 mile long New York-Washington Tunnel , this would imply an estimated cost of about 60 billion dollars. The figure could be even higher than this in view of the fact that dirt and rock needs to be lifted up instead of just moved sideways as for mountain tunnels. To put the 60 billion dollar number in perspective, it represents about $200 for every man, woman, and child in this country. Such a high cost makes the construction of such a tunnel unlikely economically although technically feasible. Alternatives such as a straight-line high speed electrically driven rail-line located tens of feet below ground level (such subways or the CERN supercollider) would seem to be a better choice economically.