DERIVATION AND PROPERTIES OF THE WITCH OF AGNESI CURVE

This is a curve studied in detail by the polymath and mathematician Marie Agnesi (1718-1799) of Milan. In its simplest form the curve reads-

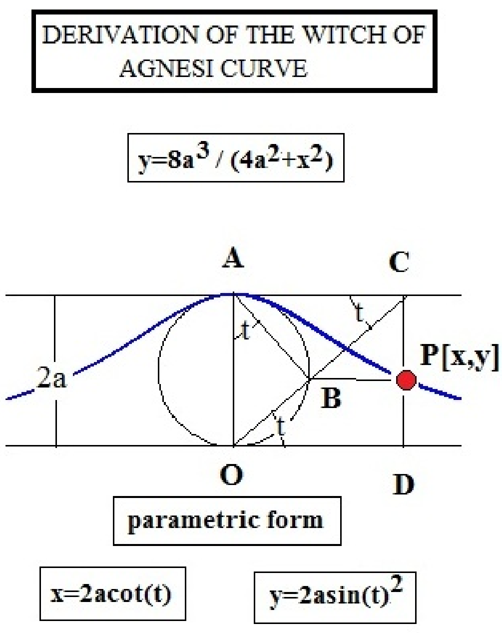
$$y = \frac{1}{(1+x^2)}$$

It is an even function of x having value of y[0]=1 and y(±∞)=0. Its integral (and hence the area under this curve) equals-

$$\int_{x=-\infty}^{+\infty} \frac{dx}{(1+x^2)} = 2\{\arctan(\infty) - \arctan(0)\} = \pi$$

The reason for the curve to be known as The Witch of Agnesi comes from a mistranslation of the Italian-Latin word versoria into English . I remember when I first ran into this curve in our analytic geometry class as a college freshman over fifty years ago, I had originally thought the name arose from the witches hat-like appearance of the curve.

We can derive the analytic version of the curve via the following graph-



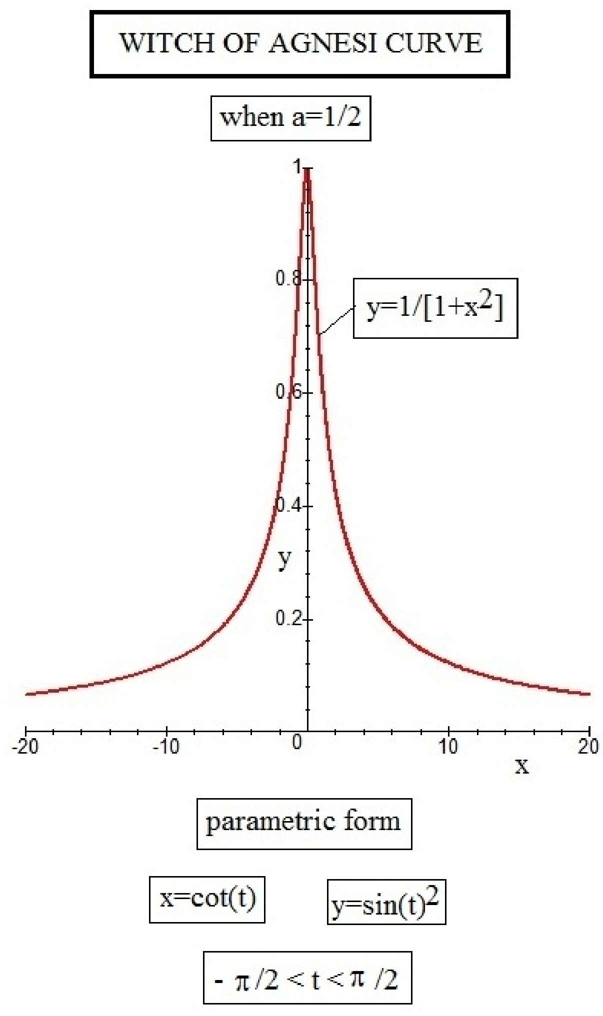
One starts with a circle of radius r=a and two horizontal parallel lines tangent to the top and bottom of the circle. Straight lines connect the points O, A, B, C, D and P as shown. The curve is defined such that point P(x,y) market as a red dot always lies on it. From the geometry one has at once that x=2a/tan(t), where the angle t is present in each of the three right triangles shown. To get y we see that y=2a-AB cos(t)=2a[1-cos(t)^2]=2asin(t)^2. We thus have the curve given in parametric form as-

$$x = 2a\cot(t) \qquad y = 2a\sin(t)^2$$

On squaring the x term and then eliminating sin(t)^2 one arrives at the Cartesian form-

$$y = \frac{(2a)^3}{[(2a)^2 + x^2]}$$

A plot of this last equation follows-



The first and second derivative of this function equals-

$$y'(x) = \frac{-16a^3 x}{(4a^2 + x^2} \qquad and \qquad y''(x) = \frac{a^3(3x^2 - 4a^2)}{(4a^2 + x^2)^3}$$

Thus the Witch of Agnesi has zero slope ar x=0 and x=±∞ and has inflection points at x=±2a/sqrt(3). The area under the entire curve equals $4\pi a^2$. That is, the area equals four times that of the generating circle.

The radius of curvature for the Agnesi curve is given by-

$$\rho(x) = \frac{[1 + y'(x)^2]^{3/2}}{y''\{x)} = \frac{(4a^2 + x^2)}{16a^3(3x^2 - 4a^2)}\sqrt{256a^8 + 512a^2x^2 + 96a^4x^4 + 16a^2x^6 + x^8}$$

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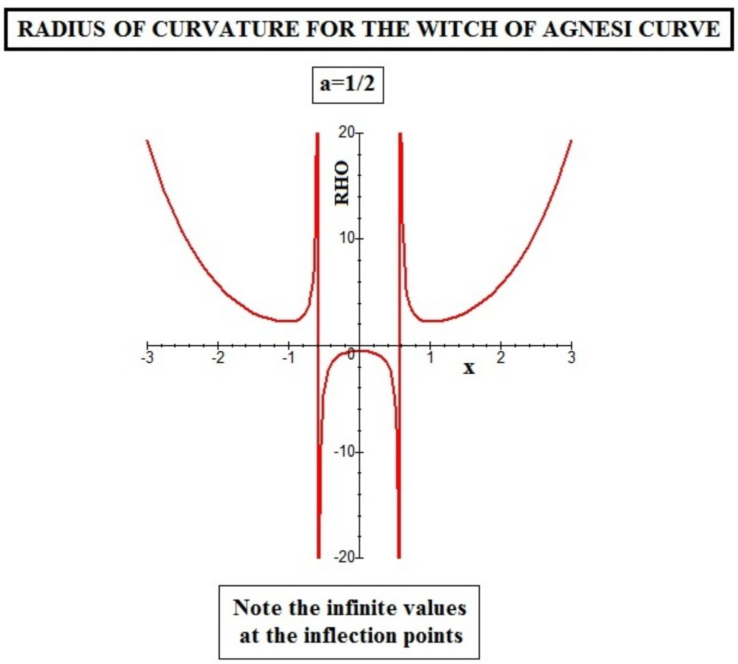
For a=1/2 it yields the following pattern-



Notice the infinite radii at x=±0.57735 corresponding to the two inflection points.

The length of the Agnesi curve extending from point A to P is given as-

$$S(x)=\int_{x=0}^{x}\sqrt{1+y'(x)^2}\,dx=\int_{t=0}^{t}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$$

This yields the rather complicated integral-

$$S(t)=2a\int_{t=0}^{t}\sqrt{[1+\cot(t)^2]^2+[2\sin(t)\cos(t)]^2}\ dt$$

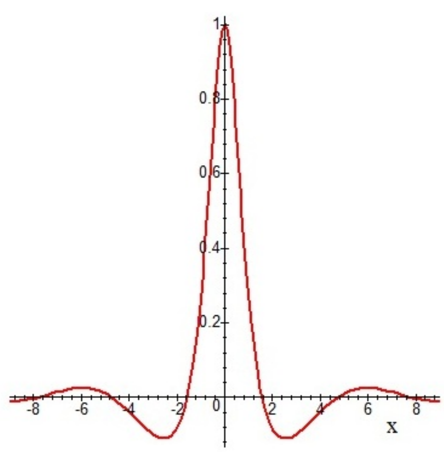
which cannot be integrated in closed form but can be evaluated numerically.

There are many variations of the Agnesi curve. For example one could consider-

$F(x)=f(x)/(1+x^2)$ with $f(x)< (1+x^2)$

So if f(x)=cos(x) we get the area under F to be π/e=1.155727. For $f(x)=\exp(-x^2)$ we get this area to be πe[1-erf(1)]. A plot of $F(x)=\cos(x)/(1+x^2)$ follows-

PLOT OF THE FUNCTION $F(X)=COS(X)/[1+X^2]$



Area under curve = π / e

Another modification is the Legendre polynomial form-

$$F(n,x)=\frac{P(n,x)}{1+x^2}\quad defined\quad in\quad -1<x<1$$

When this function is integrated over the indicated range for a given n it yields very good approximations for π , especially when n gets large. At n=40 we find-

$\pi \approx [230993148029427108414210680878530 56]/[73527402658482453321588392522327 25]$

$= 3.141592653589793238462643383278\ldots$.

, a result good to 30 decimal places.

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