SOLUTION OF THE BRAHMAGUPTA –PELL EQATION

One of the more interesting non-linear Diophantine equations is the Brahmagupta-Pell equation fist studied in detail by the Indian Mathematician Brahmagupta(598-670 AD). It reads-

$$y^2=1+Ax^2$$

or the equivalent form-

$$y=x\sqrt{A}+\frac{1}{y+x\sqrt{A}}$$

, which can be expanded into a continued fraction. Here A is any positive integer not equal to a square. One tries to solve this equation in the form $[x_n,y_n]$, neglecting the obvious lowest solution of $[x_0,y_0]=[0,1]$. Note that the name of Pell is also attached to this equation due to a translation error by Euler. The English mathematician Pell never had anything to do with the equation.

Let us begin our analysis of the B-P Equation by looking at the simplest case of A=2. Here one needs only to apply a one line computer program to generate the integer solutions. The program reads-

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for n from 1 to b do({n,sqrt(1+2x^2)]od
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We find the integer solutions through b=7 to be-

n	x_n	y_n
1	2	3
2	12	17
3	70	99
4	408	577
5	2378	3363
6	13860	19601
7	80782	114243

Looking at the ratios x_{n+1}/x_n we get 6, 5.8333, 5.82857, 5.82843, 5.828427, and 5.82842712.

So the ratio as n gets large approach the value $f=y_1+x_1$sqrt(2)=3+2sqrt(2)=5.828427125… . So we have a good idea of where x_{n+1} lies compared to x_n. The value of x_8 will lie near 80782*5.82842715=470832.002. The exact number is 470832. The ratio of y_n/x_n equals approximately sqrt(2). Thus y_8 becomes 665857. Another observation following from the table is that $x_{2n}=2y_nx_n$. . So we get as follows-

$x_2=2x_1y_1=2(2\cdot3)=12$, $x_4=2x_2y_2=2\cdot12\cdot17=408$, and $x_8=2\cdot408\cdot577=470832$.

As the next specialized form of the B-P Equation consider-

$$y^2=1+3x^3$$

Here the solution table reads-

n	x	Y
1	1	2
2	4	7
3	15	26
4	56	97
5	209	362
6	780	1351
7	2911	5042

From the table we see that f=2+1sqrt(3)=3.73205088.. and we have approximately that $x_{n+1}=f\,x_n$ and $y_n=\mathrm{sqrt}(3)x_n$. We can thus estimate that the seventh solution occurs very near $x_7=780(3.7320508)=2910.99967$ and $y_7=\mathrm{sqrt}(3)(2910.99967)=5041.99926$. The closeness of these approximate values makes it an easy job to find x_n and y_n at any larger n for a given A.

We also again have that $x_{2n}=2x_n\cdot y_n$ and $y_{2n}=\mathrm{sqrt}(1+3(x_{2n})^2)$ for A=3.Thus $x_4=2x_2\mathrm{sqrt}(1+3\cdot x_2^2)=56$.

The above exact and approximate solutions for A=2 and A=3 continue to work for all other integer A provided that A is not the square of an integer. The important information which makes the solutions possible for any positive n is the irrational number-

$$f=y_1+\mathrm{sqrt}(A)x_1$$

involving the square root of A.

Take next the special B-P Equation-

$$y^2=1+13x^2$$

Here a computer search yields the lowest non-trivial solution to be $[x_1,y_1]=[180,649]$. In this case f=649+180sqrt(13)=1297.9923. So we expect x_2 to be 233640 and y_2 to be 842401. X_3 will have the very large value-

$$x_3=f\cdot x_2=1298\cdot233640=303264720$$

As Brahmagupta already showed some 14 hundred years ago, it is possible to write the solution to his equation as-

$$x=2ab/k \text{ and } y=(b^2+13a^2)/k$$

So if we take k=1, $a=x_1$ = and $b=y_1$, we find-

$$x=x_2=233640 \quad \text{and} \quad y=y_2=842401$$

Summarizing the above , we can say , that for any A not equal to the square of an integer, we have the exact general solution-

$$\mathbf{x_{2n}=2x_ny_n \quad \text{and} \quad y_{2n}=sqrt(1+A(x_{2n})^2)=1+2A(x_n)^2}$$

exactly. To fill the gaps in x_n we use the approximation-

$$\mathbf{x_{n+1}\approx x_n[y_1+x_1sqrt(A)] \quad \text{and} \quad y_{n+1}\approx x_{n+1}sqrt(A)}$$

To test out these general results consider the case of A=40. We first do a computer search at n=1 and n=2. This produces $x_1=3$, $y_1=19$, $x_2=114$, $y_2=721$. From it we have $f=y_1+x_1sqrt(40)=37.97366596$. Next we look at $[x_3,y_3]$. By our approximation we have –

$$x_3\approx 114*f=4328.9979 \quad \text{and} \quad y_3\approx x_3sqrt(40)=27378.9867$$

This means $x_3=4329$ and $y_3=27379$. For x_4 we get $2x_2y_2=$ 164388 and

$y_4=1+2sqrt(40)114^2=1039681$. Collecting these results for A=40 produces the table-

n	x_n	y_n
1	3	19
2	114	721
3	4329	27379
4	164388	1039681

Note that this time f is quite large compared to earlier results at lower A. This results in the spacing between roots and also the intial value $[x_1,y_1]$ to become quite large. For example, we find $[x_1,y_1]=[3588,24335]$ for the particular value of A=46. I leave it to the reader to figure out how I obtained this large initial starting value.

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