MORE ON A MODIFIED PASCAL TRIANGLE

Several years ago we came up with a new modified Pascal Triangle heretofore unknown. Its first eight rows read-

1							
1	1						
1	4	1					
1	11	11	1				
1	26	66	26	1			
1	57	302	302	57	1		
1	120	1191	2416	1191	120	1	
1	247	4293	15619	15619	4293	247	1

Here the rows go from n=1 to 8, while its columns go from m=1 to n. Thus the element D[6,3]=302 and D[8,6]=4293. It is our purpose here to discuss in more detail the properties of this modified Pascal Triangle.

We begin by noting the symmetry about the vertical line containing the values 1, 4, 66, 2416. These values are given by –

$$D[2n+1, n+1]=(n+1)\{D[2n,n]+D[2n,n+1]\}.$$

So the next integer after 2416 will be 156190. Also it is noted that the sum of the elements in each row n equals n!. This means that the eighth row (n=8) will sum to 8!=40320.

Elements in the second column read 1,4,11,26,57,120,247. This means that –

$$D[n,2]=2D[n-1,2]+(n+1)$$

So the next term in the series after 247 is 502.

Further inspection of the above modified Pascal Triangle shows that any element in the nth row satisfies-

$$D[n,m]=(n+1-m)D[n-1,m-1]+(m)D[n-1,m]$$

Thus if one knows all values D(n,m] in the nth row every element D(n+1,m) will be known in the n+1 row. We have, for example, that-

D[7,3]=5D[[6,2]+3D[6,3]=1191

By playing around with the intrgers in the above modified Pascal Triangle, we have been able , after some effort, to come up with the general equality-

D[n,m]= sum((-1)^(k-1)*(n+1)!*(m+1-k)^n/((k-1)!*(n+2-k)!),k = 1 .. m)

This definition can be expressed via the one line computer program-

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D[n,m]:=sum((-1)^(k-1)*(n+1)!*(m+1-k)^n/((k-1)!*(n+2-k)!),k=1..m);
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after specifying the integer values for n and m.

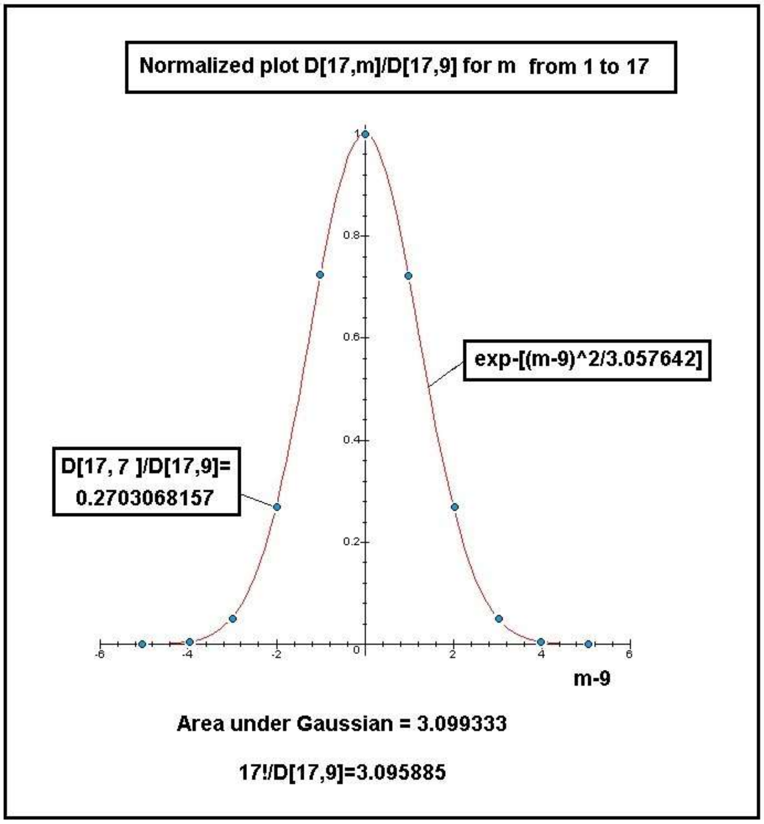
Thus we find D[10,5]=157242248 and D[15,9]=311387598411.

One notices as n gets large the value of D[n,m] for fixed n approaches the shape of a Gaussian. Here are the values for elements D[17,m] over the range $1\leq m \leq 17$ normalized to D[17,9]-

Values of D[17,m]/D[17,9] for m from 1 to 17

{1, .8703948880 10^{-14}}
{2, .1140687316 10^{-8}}
{3, .1103495517 10^{-5}}
{4, .0001294747164}
{5, .004120042934}
{6, .04976301706}
{7, .2703068157}
{8, .7236221406}
{9, 1.}
{10, .7236221406}
{11, .2703068157}
{12, .04976301706}
{13, .004120042934}
{14, .0001294747164}
{15, .1103495517 10^{-5}}
{16, .1140687316 10^{-8}}
{17, .8703948880 10^{-14}}

A point plot of these results, shown as blue circles, follows-



Superimposed on these point results is a continuous Gaussian adjusted to just two points at m=7 and m=9.The area under the Gaussian is found to be A=3.099333. The value of 17!/D[17,9]=3.095885. So the results fall very close to each other. One expects to find the area difference to become progressively smaller as n gets still larger.

We have discussed the properties of a modified Pascal Triangle where the sum of its elements in row n add up to n! As n gets large the elements along any row n approach a continuous Gaussian. These properties also allow us to relate n! to π. A one line computer program using MAPLE is also given. It allows us to quickly find any element D[n,m].

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